Homework 6 Solutions

Chapter 9

1. Let $H = \{(1), (12)\}$. Is H normal in S_3?

No, consider cosets $(13)H$ and $H(13)$. We have $(13)H = \{(13), (123)\}$ and $H(13) = \{(13), (132)\}$. Since $(13)H \neq H(13)$, H cannot be normal.

6. Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \ a, b, d \in \mathbb{R}, \ ad \neq 0 \right\}$. Is H normal in $GL(2, \mathbb{R})$?

No; consider $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \in GL(2, \mathbb{R})$. Note that $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$. Then for

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H, \text{ we have } \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \notin H.$$

Therefore H does not pass the normal subgroup test.

8. Viewing $\langle 3 \rangle$ and $\langle 12 \rangle$ as subgroups of $\mathbb{Z}$, prove that $\langle 3 \rangle / \langle 12 \rangle \cong \mathbb{Z}_4$. Similarly, prove that $\langle 8 \rangle / \langle 48 \rangle \cong \mathbb{Z}_6$. Generalize to arbitrary integers K and n.

Proof We have $\langle 3 \rangle / \langle 12 \rangle = \{0 + \langle 12 \rangle, 3 + \langle 12 \rangle, 6 + \langle 12 \rangle, 9 + \langle 12 \rangle\}$. Since $3 + \langle 12 \rangle$ has order 4, $\langle 3 \rangle / \langle 12 \rangle$ is cyclic of order 4, so $\langle 3 \rangle / \langle 12 \rangle \cong \mathbb{Z}_4$. Similarly, $\langle 8 \rangle / \langle 48 \rangle = \{0 + \langle 48 \rangle, 8 + \langle 48 \rangle, 16 + \langle 48 \rangle, 24 + \langle 48 \rangle, 32 + \langle 48 \rangle, 40 + \langle 48 \rangle\}$ and $8 + \langle 48 \rangle$ has order 6, so $\langle 8 \rangle / \langle 48 \rangle$ is cyclic of order 6 and $\langle 8 \rangle / \langle 48 \rangle \cong \mathbb{Z}_6$. ▨

In general, if $n = Km$, then $\langle K \rangle / \langle n \rangle \cong \mathbb{Z}_m$.

9. Prove that if H has index 2 in G, then $H \triangleleft G$.

Proof Let $a \in G$. If $a \in H$, then $aH = H = Ha$. If $a \notin H$, then since $|G : H| = 2$, aH is the set of elements in G that are not in H. But Ha is also the set of elements in G that are not in H, so $aH = Ha$. Since a was arbitrary, $H \triangleleft G$. ▨

12. If G is cyclic and $N \triangleleft G$, then G/N is cyclic.

Proof Assume G is cyclic. Then $\exists g \in G$ s.t. $G = \langle g \rangle$. Let $aN \in G/N$ be an arbitrary element for some $a \in G$. Then $\exists K \in \mathbb{Z}$ s.t. $a = g^K$. Thus, $aN = g^K N = (gN)^K \in \langle gN \rangle$. Hence, $G/N = \langle gN \rangle$ is cyclic. ▨

13. If G is abelian and $N \triangleleft G$, then G/N is abelian.

Proof Assume G is abelian and let $a, b \in G$. Then $(aN)(bN) = abN = baN = (bN)(aN)$. Thus, G/N is abelian. ▨

14. What is the order of the element $\overline{14} + \langle \bar{8} \rangle$ in the factor group $\mathbb{Z}_{24} / \langle \bar{8} \rangle$?

Note that $\overline{14} + \langle \bar{8} \rangle = \bar{6} + \bar{8} + \langle \bar{8} \rangle = \bar{6} + \langle \bar{8} \rangle$. Then $2 \cdot (\bar{6} + \langle \bar{8} \rangle) = \overline{12} + \langle \bar{8} \rangle = \bar{4} + \langle \bar{8} \rangle$, $3 \cdot (\bar{6} + \langle \bar{8} \rangle) = \overline{18} + \langle \bar{8} \rangle = \bar{2} + \langle \bar{8} \rangle$, and $4 \cdot (\bar{6} + \langle \bar{8} \rangle) = \overline{24} + \langle \bar{8} \rangle = \langle \bar{8} \rangle$, so $|\overline{14} + \langle \bar{8} \rangle| = 4$.

19. What is the order of the factor group $(\mathbb{Z}_{10} \oplus U(10)) / \langle (\bar{2}, \bar{9}) \rangle$?

The order of $\mathbb{Z}_{10} \oplus U(10)$ is $10 \cdot 4 = 40$ and $|(\bar{2}, \bar{9})| = \operatorname{lcm}(5, 2) = 10$, so $|(\mathbb{Z}_{10} \oplus U(10)) / \langle (\bar{2}, \bar{9}) \rangle| = \frac{40}{10} = 4$.

27. Let $G = U(16)$, $H = \{\bar{1}, \overline{15}\}$, $K = \{\bar{1}, \bar{9}\}$. Are H + K isomorphic? Are G/H + G/K isomorphic?

Since $|H| = |K| = 2$, $H \cong K \cong \mathbb{Z}_2$. Now $G/H = \{H, \bar{3} \cdot H, \bar{5} \cdot H, \bar{7} \cdot H\}$ and $G/K = \{K, \bar{3} \cdot K, \bar{5} \cdot K, \bar{7} \cdot K\}$. Note that $\bar{3} \cdot H \in G/H$ has order 4 since $(\bar{3} \cdot H)^2 = \bar{9} \cdot H = \bar{7} \cdot H$, $(\bar{3} \cdot H)^3 = (\bar{3} \cdot H)(\bar{7} \cdot H) = \overline{21} \cdot H = \bar{5} \cdot H$, $(\bar{3} \cdot H)^4 = (\bar{3} \cdot H)(\bar{5} \cdot H) = \overline{15} \cdot H = H$. Thus, $G/H \cong \mathbb{Z}_4$. However, in G/K we have $(\bar{3} \cdot K)^2 = (\bar{5} \cdot K)^2 = (\bar{7} \cdot K)^2 = K$, so $G/K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus, $G/H \not\cong G/K$.

This problem shows that $H \cong K$ does not imply $G/H \cong G/K$.

37. Let G be a finite group and let $H \triangleleft G$. Prove that the order of the element gH in G/H must divide the order of g in G.

Proof Let $|g| = n$. Then $(gH)^n = g^n H = eH = H$. Then by a result in Chapter 4, the order of gH must divide n. ▨

39. If $H \triangleleft G$, prove that $C_G(H) \triangleleft G$.

Proof Recall that $C_G(H) = \{g \in G \mid gh = hg \ \forall h \in H\}$. To apply the Normal Subgroup Test, let $g \in G$ and let $a \in C_G(H)$; we WTS that $gag^{-1} \in C_G(H)$. So let $h \in H$. Then $gag^{-1} \in C_G(H)$ iff $(gag^{-1})h = h(gag^{-1})$ iff $(gag^{-1})h(gag^{-1})^{-1} = h$. Now

$$(gag^{-1})h(gag^{-1})^{-1} = ga(g^{-1}hg)a^{-1}g^{-1} = g(g^{-1}hg)aa^{-1}g^{-1} \text{ since } H \triangleleft G \text{ and } a \in C_G(H)$$
$$= gg^{-1}hgg^{-1} = h.$$

Thus, $gag^{-1} \in C_G(H)$ and $C_G(H) \triangleleft G$. ▨

40. Let $\phi: G \to \bar{G}$ be an isomorphism. Prove that if $H \triangleleft G$, then $\phi(H) \triangleleft \bar{G}$.

Proof Let $y \in \bar{G}$ and $z \in \phi(H)$. Then $\exists h \in H$ s.t. $\phi(h) = z$. Applying the Normal Subgroup Test, we WTS that $yzy^{-1} \in \phi(H)$. Since ϕ is surjective, $\exists x \in G$ s.t. $\phi(x) = y$. Then $yzy^{-1} = \phi(x)\phi(h)\phi(x)^{-1} = \phi(xh)\phi(x^{-1}) = \phi(xhx^{-1})$. Note that $xhx^{-1} \in H$ since $H \triangleleft G$. Thus, $yzy^{-1} \in \phi(H)$, so $\phi(H) \triangleleft \bar{G}$. ▨

21. Prove that an abelian group of order 33 is cyclic.

Proof Let G be an abelian group of order 33. By Cauchy's Thrm, G has an element a of order 3 and an element b of order 11. Consider the element ab; since G is abelian, $(ab)^n = a^n b^n$. By Lagrange's Thrm, $|ab| \in \{1, 3, 11, 33\}$. If $|ab| = 1$, then $ab = e \Rightarrow b = a^{-1} \Rightarrow |a| = |b|$, a contradiction. If $|ab| = 3$, then $a^3 b^3 = e \Rightarrow b^3 = e$, contradicting $|b| = 11$. If $|ab| = 11$, then $a^{11} b^{11} = e \Rightarrow a^{11} = e \Rightarrow 3 \mid 11$, another contradiction. So $|ab| = 33$, which means $G = \langle ab \rangle$ is cyclic. ▨

31. Prove that $(\mathbb{R}^*, \cdot)$ is the internal direct product of $(\mathbb{R}^+, \cdot)$ and $\{1, -1\}$.

Proof Note first that since $(\mathbb{R}^*, \cdot)$ is abelian, every subgroup is normal. Secondly, $\mathbb{R}^+$ and $\{1, -1\}$ are both subgroups of $\mathbb{R}^*$. Certainly $\mathbb{R}^+ \cap \{1, -1\} = \{1\}$ and if $a \in \mathbb{R}^*$, a can be written as the product of an element in $\mathbb{R}^+$ and either 1 (if $a > 0$) or -1 (if $a < 0$), so $\mathbb{R}^* = \mathbb{R}^+ \cdot \{1, -1\}$. Thus, $\mathbb{R}^* = \mathbb{R}^+ \times \{1, -1\}$. ▨

49. Suppose that G is a nonabelian group of order p^3 where p is a prime, and $Z(G) \neq \{e\}$. Prove that $|Z(G)| = p$.

Proof Suppose by way of contradiction that $|Z(G)| \neq p$. By Lagrange's Thrm, $|Z(G)| \in \{1, p^2, p^3\}$. But $Z(G) \neq \{e\} \Rightarrow |Z(G)| \neq 1$, and since G is nonabelian, $Z(G) \neq G \Rightarrow |Z(G)| \neq p^3$. So $|Z(G)| = p^2$. Then $|G/Z(G)| = \frac{p^3}{p^2} = p$, so by Lagrange's Thrm $G/Z(G)$ is cyclic. Thus by Thrm 9.3, G is abelian – contradiction. So in fact $|Z(G)| = p$. ▨

50. If $|G| = pq$, where p and q are not necessarily distinct primes, prove that $|Z(G)| = 1$ or pq.

Proof If G is abelian, then $Z(G) = G \Rightarrow |Z(G)| = pq$, so assume that G is nonabelian. Then by Lagrange's Thrm, $|Z(G)| \in \{1, p, q\}$. If $|Z(G)| = p$ or q, then $|G/Z(G)| = q$ or p, respectively; then by Lagrange's Thrm $G/Z(G)$ is cyclic. Hence G is abelian by Thrm 9.3, but this is a contradiction. So in this case $|Z(G)| = 1$. ▨

55. In D_4, let $K = \{R_0, D\}$ and $L = \{R_0, D, D', R_{180}\}$. Show that $K \triangleleft L \triangleleft D_4$, but $K \ntriangleleft D_4$ (that is, normality is not transitive).

Proof Note that K is a subgroup of L of index $\frac{4}{2} = 2$. Then by #7, $K \triangleleft L$. Similarly, L is a subgroup of D_4 of index $\frac{8}{4} = 2$, so $L \triangleleft D_4$. However, the cosets $R_{90}K = \{R_{90}, H\}$ and $KR_{90} = \{R_{90}, V\}$ are not equal, so $K \ntriangleleft D_4$. ▨

56. Show that the intersection of two normal subgroups of G is a normal subgroup of G. Generalize.

Proof Let $H \triangleleft G$ and $K \triangleleft G$. Let $g \in G$ and $x \in H \cap K$. Since $x \in H$ and $H \triangleleft G$, we have $gxg^{-1} \in H$. Likewise, $x \in K$ and $K \triangleleft G$ implies $gxg^{-1} \in K$. Thus, $gxg^{-1} \in H \cap K$, so by the Normal Subgroup Test, $H \cap K \triangleleft G$. ▨

Generalization: If $H_1 \triangleleft G, H_2 \triangleleft G, \ldots, H_n \triangleleft G$, then $H_1 \cap \ldots \cap H_n \triangleleft G$.

63. If $N \lhd G$ and $|G/N| = m$, show that $x^m \in N \quad \forall x \in G$.

Proof Let $x \in G$. Since $xN \in G/N$ and $|G/N| = m$, by Lagrange's Thrm we have $(xN)^m = N$. Then $x^m N = N$ implies $x^m \in N$. ▨

65. If G is nonabelian, show that $\mathrm{Aut}(G)$ is not cyclic.

Proof Suppose by way of contradiction that $\mathrm{Aut}(G)$ is cyclic. Then by the Fund. Thrm of Cyclic Groups, $\mathrm{Inn}(G)$ is also cyclic. By Thrm 9.4, $\mathrm{Inn}(G) \cong G/Z(G)$, so this implies $G/Z(G)$ is cyclic. Then by Thrm 9.3, G is abelian – contradiction. Thus, $\mathrm{Aut}(G)$ is not cyclic. ▨

72. If $H \lhd G$ and $|H| = 2$, prove that $H \subseteq Z(G)$.

Proof Let $H = \{e, h\}$. Certainly $e \in Z(G)$, so it suffices to show that $h \in Z(G)$. Let $g \in G$. Then $gh = (ghg^{-1})g = h'g$ for some $h' \in H$ since $H \lhd G$. If $h' = e$, then $gh = h'g \Rightarrow gh = g \Rightarrow h = e$, a contradiction. Thus, $h' = h$, so $gh = hg \Rightarrow h \in Z(G)$. ▨

Chapter 10

6. Let $G = (\mathbb{R}[x], +)$, and for $f \in G$, denote by $\int f$ the antiderivative of f that passes through $(0,0)$. Show that $\int : G \to G$ is a homomorphism. What is the Kernel? Is this map a homomorphism if $\int f$ denotes the antiderivative passing through $(0,1)$?

Proof Since $\int f = F(x) + C$ passes through $(0,0)$, we have that $F(0) + C = 0 + C = 0 \Rightarrow C = 0$. So the constant of integration is 0 $\forall f \in G$ and we can safely drop it. Let $f, g \in G$. Then $\int(f+g) = F + G = \int f + \int g$, so $\int$ is a homomorphism. The Kernel is the set $\{f \in G \mid \int f = 0\} = \{0\}$, the zero function only. If $\int f$ instead denotes the antiderivative passing through $(0,1)$, then $\int f = F(x) + C \Rightarrow F(0) + C = 0 + C = 1$, so $C = 1$. Then $\int(f+g) = F + G + 1$ but $\int f + \int g = F + 1 + G + 1 = F + G + 2$, so $\int$ is <u>not</u> a homomorphism in this case. ▨

7. If $\phi : G \to H$, $\sigma : H \to K$ are homomorphisms, show that $\sigma\phi : G \to K$ is a homomorphism. How are $\mathrm{Ker}(\phi)$ and $\mathrm{Ker}(\sigma\phi)$ related? If ϕ and σ are onto and G is finite, describe $|\mathrm{Ker}(\sigma\phi) : \mathrm{Ker}(\phi)|$ in terms of $|H|$ and $|K|$.

Proof Let $a, b \in G$. Then $\sigma\phi(ab) = \sigma(\phi(ab)) = \sigma(\phi(a)\phi(b)) = \sigma(\phi(a))\sigma(\phi(b)) = \sigma\phi(a)\sigma\phi(b)$, so $\sigma\phi$ is a homomorphism. If $x \in \mathrm{Ker}(\phi)$, then $\sigma\phi(x) = \sigma(\phi(x)) = \sigma(e_H) = e_K$, so $x \in \mathrm{Ker}(\sigma\phi)$. Thus, $\mathrm{Ker}(\phi) \subseteq \mathrm{Ker}(\sigma\phi)$. Moreover, since $\mathrm{Ker}(\phi) \lhd G$ we have $\mathrm{Ker}(\phi) \trianglelefteq \mathrm{Ker}(\sigma\phi)$. Now assume ϕ, σ are onto and $|G| < \infty$. Then $\phi(G) = H$, $\sigma(H) = K$, so by the First Isomorphism Thrm, $G/\mathrm{Ker}(\phi) \cong H$, $G/\mathrm{Ker}(\sigma\phi) \cong \sigma\phi(G) = \sigma(H) = K$. Therefore, by Lagrange's Thrm $|\mathrm{Ker}(\sigma\phi) : \mathrm{Ker}\,\phi| = \dfrac{|\mathrm{Ker}(\sigma\phi)|}{|\mathrm{Ker}\,\phi|} = \dfrac{|G|/|K|}{|G|/|H|} = \boxed{\dfrac{|H|}{|K|}}$. ▨

8. Let G be a group of permutations, and for $\sigma \in G$ define $\mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd}. \end{cases}$

Prove that $\mathrm{sgn}: G \to \{1, -1\}$ is a homomorphism. What is the Kernel? Why does this allow you to conclude that $A_n \triangleleft S_n$?

Proof Let $\alpha, \beta \in G$. Recall from HW 5.19 that $\alpha\beta$ is even $\iff$ α and β are both even or both odd. If $\alpha\beta$ is even, then $\mathrm{sgn}(\alpha\beta) = 1$ and $\mathrm{sgn}(\alpha)\,\mathrm{sgn}(\beta) = 1 \cdot 1$ or $(-1)(-1)$, both of which are 1, so the property holds in this case. If $\alpha\beta$ is odd, then one of α, β is even and the other is odd, so $\mathrm{sgn}(\alpha\beta) = -1$, $\mathrm{sgn}(\alpha)\,\mathrm{sgn}(\beta) = 1(-1) = -1$. Thus sgn is a homomorphism. By construction, $\mathrm{Ker}(\mathrm{sgn}) = \{\sigma \in G \mid \sigma \text{ is even}\}$. Taking $G = S_n$ gives $\mathrm{Ker}(\mathrm{sgn}) = A_n$, and since $\mathrm{Ker}(\phi) \trianglelefteq S_n$ for every homorphism $\phi: S_n \to H$, we may conclude that $A_n \triangleleft S_n$. ▨

9. Prove that the mapping $\pi: G \oplus H \to G$ by $\pi(g, h) = g$ is a homomorphism (called the <u>projection</u> mapping). What is the Kernel?

Proof Let $g_1, g_2 \in G$, $h_1, h_2 \in H$ so that $(g_1, h_1), (g_2, h_2) \in G \oplus H$. Then

$\pi[(g_1, h_1)(g_2, h_2)] = \pi(g_1 g_2, h_1 h_2) = g_1 g_2 = \pi(g_1, h_1)\pi(g_2, h_2)$, so π is a homomorphism.

We have $\mathrm{Ker}(\pi) = \{e_G\} \oplus H$ since $\pi(g, h) = e_G$ iff $g = e_G$. ▨

15. Suppose $\phi: \mathbb{Z}_{30} \to \mathbb{Z}_{30}$ is a homomorphism and $\mathrm{Ker}(\phi) = \{\bar{0}, \overline{10}, \overline{20}\}$. If $\phi(\overline{23}) = \bar{9}$, determine all elements that map to $\bar{9}$.

By Thrm 10.1 (6), we have that $\phi^{-1}(\bar{9}) = \overline{23} + \mathrm{Ker}(\phi) = \boxed{\{\overline{23}, \bar{3}, \overline{13}\}}$.

35. Prove that $\phi: \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ given by $\phi(a, b) = a - b$ is a homomorphism. What is $\mathrm{Ker}(\phi)$? Describe the set $\phi^{-1}(3)$.

Proof Let $a, b, c, d \in \mathbb{Z}$. Then

$$\begin{aligned} \phi[(a,b) + (c,d)] &= \phi(a+c, b+d) = a + c - b - d \\ &= a - b + c - d = \phi(a,b) + \phi(c,d). \end{aligned}$$

Thus, ϕ is a homomorphism. If $(a, b) \in \mathrm{Ker}(\phi)$, then $a - b = 0 \Rightarrow a = b$. Conversely, $\phi(a, a) = a - a = 0 \Rightarrow (a, a) \in \mathrm{Ker}(\phi)$. Thus, $\mathrm{Ker}(\phi) = \{(a, a) \mid a \in \mathbb{Z}\}$.

Since $\phi(3, 0) = 3 - 0 = 3$, by Thrm 10.1 (6) we have

$$\begin{aligned} \phi^{-1}(3) &= (3, 0) + \mathrm{Ker}(\phi) \\ &= \{(a + 3, a) \mid a \in \mathbb{Z}\}. \end{aligned}$$ ▨