MAP2302 Name: Key

Exam 1 Section: ______

This exam contains 7 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	5	
2	5	
3	1	
4	2	
5	2	
6	1	
7	5	
8	4	
Total:	25	

1. (5 points) Solve the following Bernoulli equation.

$$\frac{dy}{dx} - y = e^{2x}y^2$$

$$y^{-2}\frac{dy}{dx} - y^{-1} = e^{2x}$$

Let $v = y^{-1}$

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$$

$$-\frac{dv}{dx} - v = e^{2x}$$

$$\frac{dv}{dx} + v = -e^{2x}$$

$P(x) = 1$

$$e^{\int P(x)\,dx} = e^{x}$$

$$e^{x}\frac{dv}{dx} + e^{x}v = -e^{3x}$$

$$\frac{d}{dx}[e^{x}v] = -e^{3x}$$

$$e^{x}v = -\int e^{3x}\,dx$$

$$e^{x}v = -\tfrac{1}{3}e^{3x} + C$$

$$v = -\tfrac{1}{3}e^{2x} + Ce^{-x}$$

$$y^{-1} = -\tfrac{1}{3}e^{2x} + Ce^{-x}$$

2. (5 points) Find an integrating factor and use it to solve the following equation.

$$(2xy)\,dx + (y^2 - 3x^2)\,dy = 0$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x - 2x}{2xy} = \frac{-8x}{2xy} = \frac{-4}{y} \quad \text{a function of } y$$

$$\mu(y) = \exp\left[\int -\tfrac{4}{y}\,dy\right] = y^{-4}$$

$$(2xy^{-3})\,dx + (y^{-2} - 3x^2y^{-4})\,dy = 0 \quad \text{is exact}$$

$$F(x,y) = \int 2xy^{-3}\,dx + g(y)$$

$$F(x,y) = x^2y^{-3} + g(y)$$

$$N = \frac{\partial F}{\partial y} \quad \text{so} \quad y^{-2} - 3x^2y^{-4} = -3x^2y^{-4} + g'(y)$$

$$\Rightarrow \quad g'(y) = y^{-2} \quad \text{and} \quad g(y) = -y^{-1} + C$$

$$F(x,y) = x^2y^{-3} - y^{-1} + C$$

$$\text{or} \quad x^2y^{-3} - y^{-1} = C$$

3. (1 point) Find a general solution to the given differential equation.

$$y'' - 6y' + 13y = 0.$$

Aux eqn : $r^2 - 6r + 13 = 0$

$$r = \frac{6 \pm \sqrt{36 - 4(13)}}{2}$$

$\Rightarrow r = 3 \pm 2i$

$$y(t) = C_1 e^{3t} \cos(2t) + C_2 e^{3t} \sin(2t)$$

4. (2 points) Solve the following equation.

$$y^{-1}\, dy + y e^{\cos(x)} \sin(x)\, dx = 0$$

$$-y^{-1}\, dy = y e^{\cos x} \sin x \, dx$$

$$-y^{-2}\, dy = e^{\cos x} \sin x \, dx$$

$$\int -y^{-2}\, dy = \int e^{\cos x} \sin x \, dx$$

$$y^{-1} = -e^{\cos x} + C$$

5. (2 points) Find a positive integer m such that $\varphi(x) = x^m$ is a solution to the below equation.

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = 0$$

plug in $\varphi(x) = x^m$

$$x^2 (m)(m-1)(x^{m-2}) - x(m)(x^{m-1}) - 3(x^m) = 0$$

$$m(m-1)\,x^m + (-m)\,x^m - 3x^m = 0$$

$$x^m\,(m(m-1) - m - 3) = 0$$

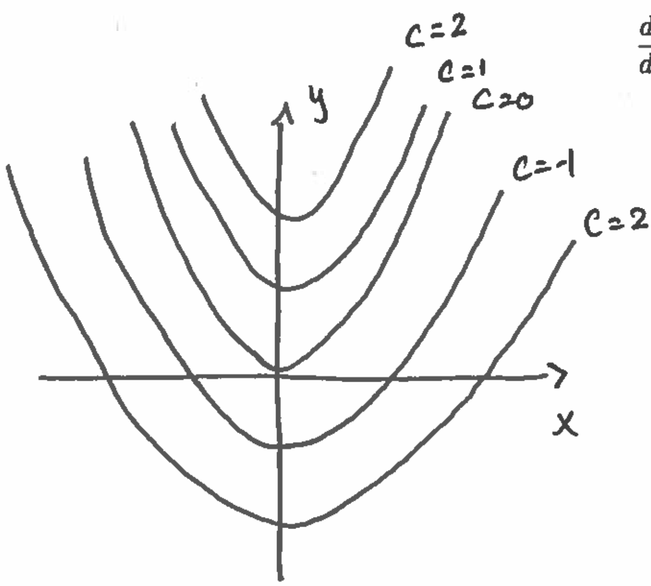
$$x^m\,(m^2 - 2m - 3) = 0$$

$$(m-3)(m+1) = 0$$

$\boxed{m = 3}$ or $m = -1$ $\longrightarrow$ if m is a solution to the quadratic then x^m is a soln.

6. (1 point) Draw the set of isoclines for the values $c = -2, -1, 0, 1, 2$ for the equation below.

$$\frac{dy}{dx} = y - x^2$$



c	isocline
-2	$x^2-2 = y$
-1	$x^2-1 = y$
0	$x^2 = y$
1	$x^2+1 = y$
2	$x^2+2 = y$

7. (5 points) Solve the following initial value problem.

$$\sin(x)\frac{dy}{dx} + \cos(x)y = x\sin(x), \qquad y\left(\frac{\pi}{2}\right) = 2$$

$$\frac{dy}{dx} + \cot x \, y = x$$

$p(x) = \cot x$

$e^{\int p(x)\,dx} = e^{\ln|\sin x|} = \sin x$ on $(0, \pi/2)$

$$\underbrace{\sin x \frac{dy}{dx} + \sin x \cot x \, y = x\sin x}$$

The problem was set up to start here

$$\Rightarrow \sin x \frac{dy}{dx} + \cos x \, y = x \sin x$$

$$\Rightarrow \frac{d}{dx}[\sin x \, y] = x\sin x$$

if this was not realized refer to steps above.

$$\sin x \, y = \int x \sin x \, dx$$

$u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$$y = \frac{1}{\sin x}\left[-x\cos x + \int \cos x \, dx\right]$$

$$y = \frac{1}{\sin x}\left[-x\cos x + \sin x + C\right]$$

$$y = -x\frac{\cos x}{\sin x} + 1 + \frac{C}{\sin x}$$

initial cond $\quad 2 = -\pi/2(0) + 1 + \frac{C}{1}$

$$\Rightarrow C = 1$$

$$\therefore \boxed{y = -x\frac{\cos x}{\sin x} + 1 + \frac{1}{\sin x}}$$

8. (4 points) Transform the following equation into a separable equation.

$$(x + y - 1)\,dx + (-x + y - 5)\,dy = 0$$

$$\left.\begin{aligned} h+k-1&=0 \\ -h+k-5&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} h+k&=1 \\ -h+k&=5 \\ \hline 2k&=6 \end{aligned}$$

$\Rightarrow k=3$ and $h=-2$

So let,

$$x = u-2 \qquad y = v+3$$
$$dx = du \qquad dy = dv$$

$$((u-2)+(v+3)-1)\,du + (-(u-2)+(v+3)-5)\,dv = 0$$

$$(u+v)\,du + (-u+v)\,dv = 0$$

$$(u+v)\,du = (u-v)\,dv$$

$$\Rightarrow \frac{dv}{du} = \frac{u+v}{u-v}$$

$$\frac{dv}{du} = \frac{1+(v/u)}{1-(v/u)}$$

Let $z = v/u$

$uz = v$

$u\frac{dz}{du} + z = \frac{dv}{du}$

$$\Rightarrow u\frac{dz}{du} + z = \frac{1+z}{1-z}$$ which is separable.