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Exam 1 Section:

This exam contains 7 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

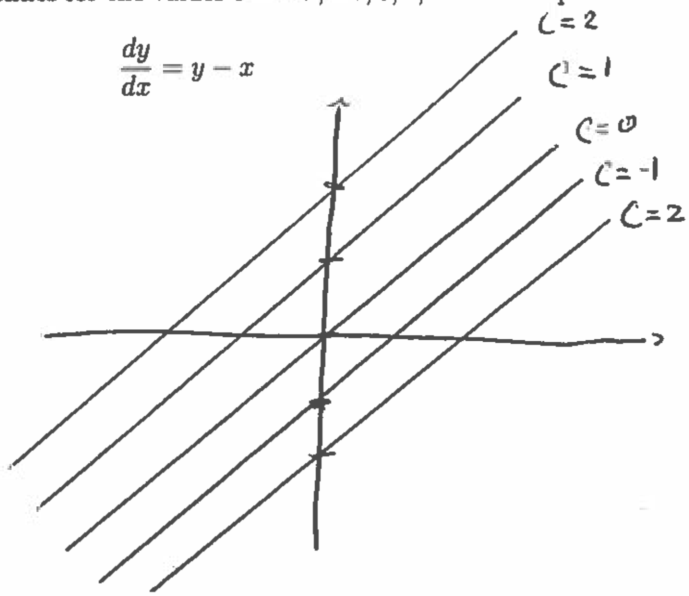
Do not write in the table to the right.

Problem	Points	Score
1	3	
2	2	
3	3	
4	3	
5	4	
6	5	
7	5	
Total:	25	

1. (a) (1 point) Draw the set of isoclines for the values $c = -2, -1, 0, 1, 2$ for the equation below.

$$\frac{dy}{dx} = y - x$$

c	isocline
-2	$y = x - 2$
-1	$y = x - 1$
0	$y = x$
1	$y = x + 1$
2	$y = x + 2$



(b) (1 point) State the two recurrence relations used in Euler's Method.

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h f(x_n, y_n)$$

(c) (1 point) Use Euler's Method to approximate the IVP's solution at $x = 1.1$.

$$\frac{dy}{dx} = x\sqrt{y}; \quad y(1) = 9$$

Let $h = 0.1$

$$y_{n+1} = 9 + (0.1)(1\sqrt{9})$$

$$y_{n+1} = 9.3$$

$$y(1.1) \approx 9.3$$

2. Suppose the following is the auxiliary equation for a homogeneous linear equation with constant coefficients.

$$(r-2)^3(r-i)^2(r+i)^2 = 0$$

(a) (1 point) Write the general solution to the differential equation.

$$y(t) = C_1 e^{2t} + C_2 t e^{2t} + C_3 t^2 e^{2t} + C_4 \cos t + C_5 t \cos t + C_6 \sin t + C_7 t \sin t$$

(b) (1 point) What is the order of the equation?

7th order

3. (3 points) Determine if the given relation is an implicit solution to the differential equation.

$$y - \ln y = x^2 + 1, \quad \frac{dy}{dx} = \frac{2xy}{y-1}$$

implicit differentiation

$$\left(\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx}\right) = 2x + 0$$

$$\frac{dy}{dx}\left(1 - \frac{1}{y}\right) = 2x$$

$$\frac{dy}{dx}\left(\frac{y-1}{y}\right) = 2x$$

$$\frac{dy}{dx} = \frac{2xy}{y-1}$$

4. (3 points) Solve the following Bernoulli equation.

$$\frac{dy}{dx} - \frac{y}{x} = x^2y^2$$

Let $\nu = y^{-1}$; $\frac{d\nu}{dx} = -1\,y^{-2}\frac{dy}{dx}$

rewrite the equation: $y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = x^2$

$$\Rightarrow \frac{d\nu}{dx} + \frac{1}{x}\nu = -x^2$$

Integrating factor

$p(x) = 1/x$; $\int p(x)dx = \ln x$

$\mu = e^{\int p(x)dx} = x$

$$x\frac{d\nu}{dx} + \nu = -x^3$$

so $\frac{d}{dx}[x\nu] = -x^3$

Hence $x\nu = \int -x^3 dx + C$

$$x\nu = -\frac{x^4}{4} + C$$

so $\boxed{y^{-1} = \frac{-x^3}{4} + Cx^{-1}}$

5. (4 points) Solve the following equation.

$$\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin(\theta)}{y^2+1}$$

$$\frac{y^2+1}{y}\,dy = \theta\sin\theta\,d\theta$$

$$\Rightarrow \int y + \frac{1}{y}\,dy = \int \theta\sin\theta\,d\theta$$

I.B.P

$u=\theta$ $dv = \sin\theta\,d\theta$

$du = d\theta$ $v = -\cos\theta$

$$\Rightarrow \frac{y^2}{2} + \ln y = -\theta\cos\theta + \int\cos\theta\,d\theta$$

$$\boxed{\frac{y^2}{2} + \ln y = -\theta\cos\theta + \sin\theta + C}$$

6. (5 points) Find an integrating factor and use it to solve the following equation.

$$(y^2 + 2xy)\,dx - x^2\,dy = 0$$

$\frac{\partial M}{\partial y} = 2y + 2x \qquad \frac{\partial N}{\partial x} = -2x \qquad$ Not exact

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -\frac{(2y + 2x + 2x)}{y(y+2x)}$$

$$= \frac{-(2y + 4x)}{y(y+2x)}$$

$$= \frac{-2\,(y+2x)}{y(y+2x)}$$

$= -2/y \quad$ a function of y

Integrating factor:

$$\mu(y) = e^{\int 2/y \, dy} = y^{-2}$$

New equation: $(1 + 2xy^{-1})dx + (-x^2)y^{-2}dy = 0$

$$F(x,y) = \int -x^2y^{-2}\,dy + h(x)$$

$$F(x,y) = x^2y^{-1} + h(x)$$

$$\frac{\partial F}{\partial x} = 2xy^{-1} + h'(x) = 1 + 2xy^{-1}$$

so $h'(x) = 1$ and $h(x) = x$

$$\boxed{\text{Soln: } x^2y^{-1} + x = C}$$

7. (5 points) Solve the following initial value problem.

$$t^2\frac{dx}{dt} + 3tx = t^4 \ln t + 1, \quad x(1) = 0$$

$$\frac{dx}{dt} + \frac{3}{t}x = t^2 \ln t + t^{-2} \qquad \mu(t) = e^{3\int \frac{1}{t}dt} = t^3$$

$$t^3\frac{dx}{dt} + 3t^2 x = t^5 \ln t + t$$

$$\frac{d}{dx}[t^3 x] = t^5 \ln t + t$$

IBP

$u = \ln t \quad dv = t^5 dt$

$du = \frac{1}{t}dt \quad v = \frac{1}{6}t^6$

$$t^3 x = \int t^5 \ln t + t \, dt + C$$

$$t^3 x = \left[\frac{1}{6}t^6 \ln t - \int \frac{1}{6} t^6 t^{-1} dt + \frac{t^2}{2} + C\right]$$

$$t^3 x = \frac{1}{6}t^6 \ln t - \frac{1}{36}t^6 + \frac{t^2}{2} + C$$

$$x = \frac{1}{6}t^3 \ln t - \frac{1}{36}t^3 + \frac{t^{-1}}{2} + Ct^{-3}$$

at $t=1$, $x=0$

$$0 = -\frac{1}{36} + \frac{1}{2} + C$$

$$C = \frac{-17}{36}$$

$$x = \frac{1}{6}t^3 \ln t - \frac{1}{36}t^3 + \frac{t^{-1}}{2} - \frac{17}{36}t^{-3}$$