MAP2302 Name: Key

Practice Exam 2 Section: 4D15

This exam contains 6 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	2	
2	4	
3	5	
4	3	
5	3	
6	5	
7	3	
Total:	25	

1. Write the form of the particular solution for the following differential equations.
   DO NOT find the coefficients.

   (a) (1 point) $2y'' - y' + 6y = t^2 e^{-t}\sin(t)$

   $2r^2 - r + 6 = 0$ $\quad y_p = (A_2t^2 + A_1t + A_0)\,e^{-t}\sin(t) + (B_2t^2 + B_1t + B_0)\,e^{-t}\cos t$

   $r = \frac{1 \pm i\sqrt{47}}{4}$

   (b) (1 point) $y'' - 4y = e^{2t} + te^{2t} + t^2e^{2t}$

   $r^2 - 4 = 0$ $\quad y_p = t\,(A_2t^2 + A_1t + A_0)e^{2t}$

   $r = \pm 2$

2. (4 points) Use the method of undetermined coefficients to find the general solution.

$$y'' - 3y' + 2y = \sin t$$

$r^2 - 3r + 2 = 0$ $\quad y_1 = e^{2t},\ y_2 = e^t$

$(r-2)(r-1) = 0$

$r = 2 \quad r = 1$

$y_p = A\sin t + B\cos t$

$y_p' = A\cos t - B\sin t$

$y_p'' = -A\sin t - B\cos t$

$y_p'' - 3y_p' + 2y_p = (-A\sin t - B\cos t) - 3(A\cos t - B\sin t) + 2(A\sin t + B\cos t) = \sin t$

$\Rightarrow (A + 3B)\sin t + (B - 3A)\cos t = \sin t$

$A + 3B = 1$

$B - 3A = 0$

$\Rightarrow A = 1/10,\ B = 3/10$

$y_g = y_p + C_1y_1 + C_2y_2 = \frac{1}{10}\sin t + \frac{3}{10}\cos t + C_1e^{2t} + C_2e^{t}$

3. Consider the following differential equation.

$$t^2y'' + 5ty' + 3y = t$$

(a) (1 point) Find the homogeneous solutions to the above differential equation.

$ar^2 + (b-a)r + c = 0$ $\quad r_1 = -1$, $r_2 = -3$ are roots

$r^2 + 4r + 3 = 0$ $\quad y_1 = t^{-1}$, $y_2 = t^{-3}$

(b) (1 point) Calculate the Wronskian of the homogenous solutions.

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1'$$
$$= t^{-1}(-3t^{-4}) - (t^{-3})(-t^{-2})$$
$$= -3t^{-5} + t^{-5}$$
$$= -2t^{-5}$$

(c) (2 points) Find the particular solutions to the above differential equation.

$$V_1 = -\int \frac{f(t)\, y_2}{W[y_1, y_2]}\, dt = -\int \frac{t^5}{(-2)\, t^3\, t}\, dt = \int \frac{t}{2}\, dt = \frac{1}{4} t^2$$

$$V_2 = \int \frac{f(t)\, y_1}{W[y_1, y_2]}\, dt = \int \frac{t^5}{(-2)\, t\, t}\, dt = \int -\frac{t^3}{2}\, dt = \frac{-1}{8} t^4$$

Note: $f(t) = 1/t$

$$y_p = V_1 y_1 + V_2 y_2 = \frac{1}{4} t^2 (t^{-1}) + \left(\frac{-1}{8} t^4\right)(t^{-3}) = \frac{1}{8} t$$

(d) (1 point) Write the general solution to the above differential equation.

$$y_g = y_p + C_1 y_1 + C_2 y_2$$
$$y_g = \frac{1}{8} t + C_1 t^{-1} + C_2 t^{-3}$$

4. Suppose homogeneous differential equation has the following auxiliary equation.

$$(r-2)^3(r-5)(r-7)(r-4i))^2(r+4i))^2 = 0$$

(a) (2 points) Write a fundamental solution set for the differential equation.

$$\left\{ e^{2t},\ te^{2t},\ t^2e^{2t},\ e^{5t},\ e^{7t},\ \cos(4t),\ \sin(4t),\ t\cos(4t),\ t\sin(4t) \right\}$$

(b) (1 point) What is the order of the original differential equation?

9

5. (3 points) Find a particular solution to

$$y'' - 6y' + 9y = t^{-3}e^{3t}$$

by using variation of parameters.

$$r^2 - 6r + 9 = 0$$

$$\Rightarrow (r-3)^2 = 0$$

$r = 3$ double root

$$y_1 = e^{3t}, \quad y_2 = te^{3t}$$

$$W[y_1, y_2] = e^{3t}(3te^{3t} + e^{3t}) - te^{3t}(3e^{3t})$$

$$= \cancel{3te^{6t}} + e^{6t} - \cancel{3te^{6t}}$$

$$= e^{6t}$$

$$V_1 = -\int \frac{e^{3t}\, te^{3t}}{t^3(e^{6t})}\, dt = -\int \frac{1}{t^2}\, dt = t^{-1}$$

$$V_2 = \int \frac{e^{3t}e^{3t}}{t^3(e^{6t})}\, dt = \int \frac{1}{t^3}\, dt = -\frac{1}{2}t^{-2}$$

$$y_p = V_1 y_1 + V_2 y_2$$

$$\boxed{y_p = (t^{-1})(e^{3t}) + \left(-\tfrac{1}{2}t^{-2}\right)(te^{3t})}$$

6. (5 points) Use Laplace transformations to solve the initial value problem.

$$y'' + 6y' + 5y = 12e^t; \quad y(0) = -1, \quad y'(0) = 7$$

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\{12e^t\}$$

$$(s^2 Y(s) + s - 7) + 6(sY(s) + 1) + 5Y(s) = \frac{12}{s-1}$$

$$Y(s)(s^2 + 6s + 5) + s - 1 = \frac{12}{s-1}$$

$$Y(s)(s^2 + 6s + 5) = \frac{12 - (s-1)^2}{(s-1)}$$

$$Y(s) = \frac{-s^2 + 2s + 11}{(s-1)(s^2+6s+5)}$$

$$Y(s) = \frac{-s^2 + 2s + 11}{(s-1)(s+5)(s+1)} = \underbrace{\frac{A}{(s-1)} + \frac{B}{(s+5)} + \frac{C}{(s+1)}}_{\text{partial fraction decomp}} = \frac{A(s+5)(s+1) + B(s-1)(s+1) + C(s-1)(s+5)}{(s-1)(s+5)(s+1)}$$

plugging in $s = 1$, $s = -5$, $s = -1$ into $-s^2 + 2s + 11 = A(s+5)(s+1) + B(s-1)(s+1) + C(s-1)(s+5)$

We get $A = 1$, $B = 1$, $C = -1$

$$Y(s) = \frac{1}{s-1} + \frac{1}{s+5} + \frac{-1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$y(t) = e^t + e^{-5t} + e^{-t}$$

7. (3 points) Find $\mathscr{L}^{-1}\{F\}(t)$.

i) $F(s) = \dfrac{s^2 - 26s - 47}{(s-1)(s+2)(s+5)}$

ii) $F(s) = \dfrac{-2s^2 - 3s - 2}{s(s+1)^2}$

(i) $\dfrac{s^2-26s-47}{(s-1)(s+2)(s+5)} = \dfrac{A}{(s-1)} + \dfrac{B}{(s+2)} + \dfrac{C}{s+5}$

$\Rightarrow \dfrac{s^2-26s\ 47}{(s-1)(s+2)(s+5)} = \dfrac{A(s+2)(s+5) + B(s-1)(s+5) + C(s-1)(s+1)}{(s-1)(s+2)(s+5)}$

$\Rightarrow s^2 - 26s - 47 = A(s+2)(s+5) + B(s-1)(s+5) + C(s-1)(s+1)$

plugging in $s = 1$, $s = -2$, $s = -5$ we get

$A = -4$, $B = -1$, $C = 6$

$\Rightarrow F(s) = \dfrac{-4}{s-1} - \dfrac{1}{s+2} + \dfrac{6}{s+5} \Rightarrow \boxed{\mathcal{L}^{-1}\{F\} = -4e^{t} - 1e^{-2t} + 6e^{-5t}}$

(ii) $\dfrac{-2s^2-3s-2}{s(s+1)^2} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s}$

$\Rightarrow -2s^2 - 3s - 2 = A\,s(s+1) + Bs + C(s+1)^2$

plugging in $s = -1$, $s = 0$, and $s = 1$ we get $B = 1$, $C = -2$, $A = 0$

$F(s) = \dfrac{1}{(s+1)^2} + \dfrac{-2}{s}$ $\boxed{\mathcal{L}^{-1}\{F\} = e^{-t}t - 2}$