MAP2302 **Name:** Soln.

Exam 2 **Section:**

This exam contains 7 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	2	
2	3	
3	6	
4	2	
5	4	
6	5	
7	3	
Total:	25	

1. Write the form of the particular solution for the following differential equations. DO NOT find the coefficients.

(a) (1 point) $y'' + 5y' + 6y = \sin(t) - \cos(2t)$

Auxeqn: $r^2+5r+6=0$

$(r+2)(r+3)=0$

$r=-2, -3$

$\therefore\ s=0$

$y_p = A\sin t + B\cos t + C\sin(2t) + D\cos(2t)$

(b) (1 point) $y'' - 4y' + 4y = t^2e^{2t} - e^{2t}$

Auxeqn: $r^2-4r+4=0$

$(r-2)^2=0$

$r=2$ doubleroot

$\therefore\ s=2$

$y_p = t^2(At^2+Bt+C)e^{2t}$

2. (3 points) Use the method of undetermined coefficients to find the general solution.

$$y'' + 2y' + y = 3t^2$$

Auxeqn: $r^2+2r+1=0$

$(r+1)^2=0$

$r=-1$ doubleroot

$s=0$

$y_p = At^2+Bt+C$

$y_p' = 2At+B$

$y_p'' = 2A$

$2A + 2(2At+B) + At^2+Bt+C = 3t^2$

$At^2 + (4A+B)t + (2A+2B+C) = 3t^2$

$A=3$

$12+B=0 \Rightarrow B=-12$

$6-24+C=0 \Rightarrow C=18$

$y_s = 3t^2-12t+18 + C_1e^{-t} + C_2te^{-t}$

3. (6 points) Find the general solution to the following equation for $t > 0$.

$$t^2y'' + 7ty' - 7y = t^3$$ Cauchy Euler

Auxeqn: $r^2 + (7-1)r - 7 = 0$

$r^2 + 6r - 7 = 0$

$(r-1)(r+7) = 0$

$r = 1, -7$

Homogeneous solns: $y_1 = t \quad y_2 = t^{-7}$

$$W[y_1, y_2] = \begin{vmatrix} t & t^{-7} \\ 1 & -7t^{-8} \end{vmatrix} = -7t^{-7} - t^{-7} = -8t^{-7}$$

Standard Form: $y'' + \frac{7}{t}y' - \frac{7}{t^2}y = t$

$V_1 = -\int \frac{t\, t^{-7}}{-8t^{-7}}\, dt \qquad V_2 = \int \frac{t\, t}{-8t^{-7}}\, dt$

$V_1 = \frac{1}{8}\int t\, dt \qquad V_2 = -\frac{1}{8}\int t^9\, dt$

$V_1 = \frac{t^2}{16} \qquad V_2 = -\frac{1}{80}t^{10}$

$$y_p = V_1y_1 + V_2y_2 = \frac{t^2}{16}(t) + \left(-\frac{1}{80}\right)t^{10}t^{-7} = \left(\frac{1}{16} - \frac{1}{80}\right)t^3 = \frac{t^3}{20}$$

$$\boxed{y_g = \frac{t^3}{20} + C_1 t + C_2 t^{-7}}$$

4. (a) (1 point) Given the fundamental solution set and the particular solution determine the general solution.

$$y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2$$

$$y_p = x^2 \qquad \{e^x,\ e^{-x}\cos(2x),\ e^{-x}\sin(2x)\}$$

$y_p = x^2 + C_1 e^x + C_2 e^{-x}\cos(2x) + C_3 e^{-x}\sin(2x)$

(b) (1 point) Let

$$L[y] := y''' + y' + xy; \qquad y_1(x) := \sin(x),\ y_2(x) := x.$$

Given

$$L[y_1] = x\sin(x), \qquad L[y_2] = x^2 + 1,$$

find a solution to

$$L[y] = 2x\sin(x) - x^2 - 1. = 2L[y_1] - L[y_2]$$

$\therefore y = 2y_1 - y_2$

$\boxed{y = 2\sin x - x}$

5. (4 points) Given the two linearly independent solutions to the corresponding homogeneous equation find a particular solution to the equation.

$$ty'' + (1-2t)y' + (t-1)y = te^t; \qquad y_1 = e^t,\ y_2 = e^t\ln(t)$$

Standard form: $y'' + \frac{(1-2t)}{t}y' + \frac{(t-1)}{t}y = e^t$

$$W[y_1, y_2] = \begin{vmatrix} e^t & e^t\ln t \\ e^t & \frac{e^t}{t} + e^t\ln t \end{vmatrix} = \frac{e^{2t}}{t} + e^{2t}\ln t - e^{2t}\ln t$$

$$W[y_1, y_2] = \frac{e^{2t}}{t}$$

$$V_1 = -\int \frac{t\,e^t e^t \ln t}{e^{2t}}\,dt \qquad V_2 = \int \frac{t\,e^t e^t}{e^{2t}}\,dt$$

I.B.P
$u = \ln t \quad dv = t\,dt$
$du = \frac{1}{t}dt \quad v = \frac{t^2}{2}$

$$V_1 = -\int t\ln t\,dt \qquad V_2 = \int t\,dt$$

$$V_1 = -\left[\frac{t^2}{2}\ln t - \int \frac{t}{2}dt\right] \qquad V_2 = \frac{t^2}{2}$$

$$V_1 = -\left[\frac{t^2}{2}\ln t - \frac{1}{4}t^2\right]$$

$$V_1 = -\frac{t^2}{2}\ln t + \frac{t^2}{4}$$

$$y_p = V_1y_1 + V_2y_2 = \left(-\frac{t^2}{2}\ln t + \frac{t^2}{4}\right)e^t + \frac{t^2}{2}\left(e^t\ln t\right)$$

$$\boxed{y_p = \frac{e^t t^2}{4}}$$

6. (5 points) Use Laplace transformations to solve the initial value problem.

$$y'' - 4y' + 5y = 0; \quad y(0) = 2, \ y'(0) = 7$$

Let $Y = \mathcal{L}\{y\}$

$$[s^2 Y - 2s - 7] - 4[sY - 2] + 5Y = 0$$

$$(s^2 - 4s + 5) Y = 2s - 1$$

$$[(s-2)^2 + 1] Y = 2s - 1$$

$$Y = \frac{2s-1}{(s-2)^2+1} = \frac{2s-4+3}{(s-2)^2+1} = \frac{2(s-2)}{(s-2)^2+1} + \frac{3(1)}{(s-2)^2+1}$$

$$\boxed{y = \mathcal{L}^{-1}\{Y\} = 2e^{2t}\cos t + 3e^{2t}\sin t}$$

7. (a) (2 points) Find the inverse Laplace transform of the following function.

$$F(s) = \frac{s^2 - s + 1}{(s+1)(s^2 - 3s + 2)}$$

$$F(s) = \frac{s^2 - s + 1}{(s+1)(s-2)(s-1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$\therefore \; s^2 - s + 1 = A(s-2)(s-1) + B(s+1)(s-1) + C(s+1)(s-2)$$

$(s=1)$ $\quad 1 - 1 + 1 = 0 + 0 + C(-2) \quad \Rightarrow C = -\frac{1}{2}$

$(s=2)$ $\quad 4 - 2 + 1 = 0 + 3B + 0 \quad \Rightarrow B = 1$

$(s=-1)$ $\quad 1 + 1 + 1 = 6A + 0 + 0 \quad \Rightarrow A = \frac{1}{2}$

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{1}{s+1}\right) + \left(\frac{1}{s-2}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right)\right\}$$

$$\mathcal{L}^{-1}\{F\} = \frac{1}{2}e^{-t} + e^{2t} - \frac{1}{2}e^{t}$$

(b) (1 point) Determine $\mathcal{L}\{t\cos(bt)\}$ where b is a constant.

$$\mathcal{L}\{t\cos(bt)\} = -\frac{d}{ds}\left\{\frac{s}{s^2+b^2}\right\}$$

$$\mathcal{L}\{t\cos(bt)\} = -\left[\frac{(s^2+b^2)(1) - s(2s)}{(s^2+b^2)^2}\right]$$

$$\mathcal{L}\{t\cos(bt)\} = -\left[\frac{b^2 - s^2}{(s^2+b^2)^2}\right]$$

$$\mathcal{L}\{t\cos(bt)\} = \frac{s^2 - b^2}{(s^2+b^2)^2}$$