Practice Exam 3

① $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$.

$|G| = |\mathbb{Z}_4||\mathbb{Z}_2||\mathbb{Z}_5| = 4 \cdot 2 \cdot 5 = \boxed{40}$

G is not cyclic. By a Thm from Ch. 8, $G_1 \oplus \cdots \oplus G_n$ is cyclic iff $\gcd(|G_i|, |G_j|) = 1 \; \forall i \neq j$. But $\gcd(4,2) = 2 \neq 1$, so $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$ cannot be cyclic.

② $G = \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7$

How many elts of order 2? Each of the factors $\mathbb{Z}_8$, $\mathbb{Z}_4$, + $\mathbb{Z}_2$ has an element of order 2. So there are $2^3 - 1 = 7$ elements of order 2.

[They are $(\bar{4},\bar{0},\bar{0},\bar{0},\bar{0})$, $(\bar{0},\bar{2},\bar{0},\bar{0},\bar{0})$, $(\bar{0},\bar{0},\bar{1},\bar{0},\bar{0})$, $(\bar{4},\bar{2},\bar{0},\bar{0},\bar{0})$, $(\bar{4},\bar{0},\bar{1},\bar{0},\bar{0})$, $(\bar{0},\bar{2},\bar{1},\bar{0},\bar{0})$, and $(\bar{4},\bar{2},\bar{1},\bar{0},\bar{0})$.]

Does it have any elements of order 14? Yes, $(\bar{0},\bar{0},\bar{1},\bar{0},\bar{1})$ has order $\operatorname{lcm}(1,1,2,1,7) = 14$.

③ If G, H are groups, f is a homomorphism from G to H if $f: G \to H$ is a map s.t. $\forall g_1, g_2 \in G$, we have $f(g_1 g_2) = f(g_1) f(g_2)$.

④ State the First Isomorphism Theorem for groups.

Thm Let $\phi: G \to H$ be a group homomorphism. Then $G/\mathrm{Ker}(\phi) \cong \phi(G)$.

⑤ Let $f: D_4 \to \mathbb{Z}_4$ be defined by $f(x) = \begin{cases} \bar{0} & \text{if } x \text{ is a rotation} \\ \bar{2} & \text{if } x \text{ is not a rotation} \end{cases}$

Prove that f is a group homomorphism, but not an isomorphism.

Proof Let $x, y \in D_4$. If x + y are both rotations or both reflections, then xy is a rotation and we have $f(xy) = \bar{0} = \begin{cases} \bar{0} + \bar{0} \\ \bar{2} + \bar{2} \end{cases} = f(x) + f(y)$. If one of x + y is a rotation, say x, and y is a reflection, then xy is a reflection and $f(xy) = \bar{2} = \bar{0} + \bar{2} = f(x) + f(y)$. Thus, f is a homomorphism.

Since $\mathrm{Im}(f) = \{\bar{0}, \bar{2}\} \neq \mathbb{Z}_4$, f is not onto, and thus f is not an isomorphism.

(Alternatively, $f(R_0) = f(R_{90}) = \bar{0}$ but $R_0 \neq R_{90} \Rightarrow f$ is not injective.) ∎

⑥ With f as in #5 find (a) $\mathrm{Im}(f)$ and (b) $\mathrm{Ker}(f)$.

(a) $\mathrm{Im}(f) = \{\bar{0}, \bar{2}\} = \boxed{\langle \bar{2} \rangle}$

(b) $\mathrm{Ker}(f) = \{x \in G \mid f(x) = \bar{0}\} = \{R_0, R_{90}, R_{180}, R_{270}\}$
$= \boxed{\langle R_{90} \rangle}$.