MAC2313 Test 3 A Fall 2015

(5 pts) 1. Let D be the solid which lies below the ellipsoid $(x/2)^2 + (y/4)^2 + (z/\sqrt{2})^2 = 1$ and above the plane $z = 1$; what is the projection of the solid onto the x, y-plane?

$z = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 + \frac{1}{2} = 1$

$\frac{x^2}{4} + \frac{y^2}{16} = \frac{1}{2}$

$\frac{x^2}{2} + \frac{y^2}{8} = 1$

A. The ellipse $x^2/2 + y^2/8 = 1$

B. The ellipse $(x/2)^2 + (y/4)^2 = 1$

C. The ellipse $x^2/8 + y^2/32 = 1$

D. The circle $x^2 + y^2 = (2\sqrt{2})^2$

E. none of the above

(5 pts) 2. If a point in Cartesian coordinates is given by $(x, y, z) = (-1, 1, \sqrt{2})$, then how many of the following are true when the point is expressed in spherical coordinates?

QII

i. $\rho = 2$ ✓

$\rho = \sqrt{(-1)^2 + 1^2 + \sqrt{2}^2} = \sqrt{1+1+2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}, \frac{7\pi}{4}$

ii. $\theta = -\pi/4$ ✗

iii. $\phi = \pi/4$ ✓

$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

iv. $x^2 + y^2 = \rho^2 \sin^2 \phi$ ✓

$x^2 + y^2 = \rho^2 \sin^2\phi\,(\cos^2\theta + \sin^2\theta) = \rho^2 \sin^2\phi$

A. 0 B. 1 C. 2 D. 3 E. 4

(5 pts) 3. The integral $\int_1^2 \int_z^{2z} \int_1^3 4y \; dx \; dy \; dz$ is equal to?

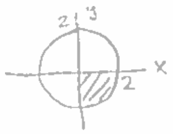
A. 48 B. 24 C. 28 D. 36 E. 3

$\int_1^2 \int_z^{2z} 4yx\Big|_{x=1}^{3} dy\,dz = 8\int_1^2 \int_z^{2z} y\,dy\,dz = 4\int_1^2 y^2\Big|_z^{2z} dz = 4\int_1^2 (4z^2 - z^2)\,dz$

$= 4\int_1^2 3z^2\,dz = 4\,z^3\Big|_1^2 = 4(8-1) = 4(7) = 28$

(5 pts) 4. Let $D = \{(x,y) \mid 0 \le x \le 2,\ -\sqrt{4-x^2} \le y \le 0\}$, then $\int\int_D\ y\ dA$ is equal to:

A. 0   B. $-4/3$ C. $-16/3$ D. $-1/2$ E. $-8/3$



$$\int_{3\pi/2}^{2\pi}\int_0^2 r^2\sin\theta\,dr\,d\theta = \int_{3\pi/2}^{2\pi}\frac{r^3}{3}\Big|_0^2 \sin\theta\,d\theta = \frac{8}{3}\int_{3\pi/2}^{2\pi}\sin\theta\,d\theta = -\frac{8}{3}\cos\theta\Big|_{3\pi/2}^{2\pi}$$

$$= -\frac{8}{3}(1-0) = -\frac{8}{3}$$

(5 pts) 5. Let $f(x,y)$ and $g(x,y)$ be continuous functions over a closed and bounded planar region R and let $h(x,y,z)$ be continuous on a closed and bounded solid D. How many of the following are necessarily true?

i. The volume of the solid D is given by $\int\int\int_D\ 1\ dV$. ✓

ii. $\int\int_R\ g(x,y)\ dA \ge 0$. ×

iii. The average value of f on R is given by $[\int\int_R\ f(x,y)\ dA]/[\int\int_R\ 1\ dA]$. ✓

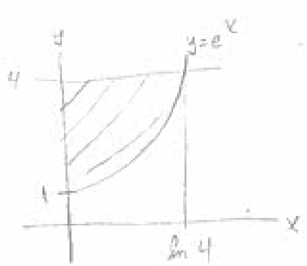
iv. The volume between f and g over R is given by $\int\int_R\ g(x,y) - f(x,y)\ dA$. ×   could be f-g

A. 0   B. 1   C. 2   D. 3   E. 4

(5 pts) 6. The integral $\int_0^{\ln 4}\int_{e^x}^4\ g(x,y)\ dy\ dx$ is equal to:

A. $\int_1^4\int_0^{\ln y} g(x,y)\ dx\ dy$ B. $\int_1^4\int_1^{\ln y} g(x,y)\ dx\ dy$ C. $\int_0^4\int_{\ln y}^1 g(x,y)\ dx\ dy$ D. $\int_0^4\int_1^{\ln y} g(x,y)\ dx\ dy$

E. none of the above



left bound $x = 0$
right bound $x = \ln y$

$\int_1^4\int_0^{\ln y} g(x,y)\,dx\,dy$

(5 pts) 7. The area which lies between the curves $y = 4 - x^2$ and $y = 2x + 1$ is equal to:

$4 - x^2 = 2x + 1 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow (x+3)(x-1) = 0 \Rightarrow x = -3, 1$

A. 0   B. 17/3   C. 22/3   D. 52/3   E. 32/3

$A = \int_{-3}^{1} \left[(4-x^2) - (2x+1)\right] dx = \int_{-3}^{1} (-x^2 - 2x + 3)\, dx = -\frac{x^3}{3} - x^2 + 3x \Big|_{-3}^{1} = \left[-\frac{1}{3} - 1 + 3\right] + \left[\cancel{9 - 9} + 9\right]$

$= -\frac{1}{3} + 2 + 9 = 11 - \frac{1}{3} = \frac{32}{3}$

(5 pts) 8. Let D be the solid bounded by the surfaces $z = 6 - x^2 - y^2$ and $z = 4$, the integral $\int\int\int_D x^2 y \, dV$ expressed in cylindrical coordinates is given by: $6 - x^2 - y^2 = 4 \Rightarrow x^2 + y^2 = 2$

$x^2 y = (r\cos\theta)^2 r \sin\theta = r^3 \sin\theta \cos^2\theta \cdot r = r^4 \sin\theta \cos^2\theta$

A. $\int_0^{2\pi} \int_0^{2} \int_4^{6-r^2} r^4 \sin^2\theta \cos\theta \, dz\, dr\, d\theta$

B. $\int_0^{\pi} \int_0^{\sqrt{2}} \int_4^{6-r^2} r^3 \cos^2\theta \sin\theta \, dz\, dr\, d\theta$

C. $\int_0^{2\pi} \int_0^{\sqrt{2}} \int_4^{6-r^2} r^4 \sin\theta \cos\theta \, dz\, dr\, d\theta$

D. $\int_0^{2\pi} \int_0^{\sqrt{2}} \int_4^{6-r^2} r^4 \sin\theta \cos^2\theta \, dz\, dr\, d\theta$

E. none of the above

(5 pts) 9. Let D be a planar region defined by $D = \{\ (r, \theta) \mid 1 \le r \le 2,\ \pi/2 \le \theta \le 3\pi/2\ \}$; this region expressed in Cartesian coordinates is given by:

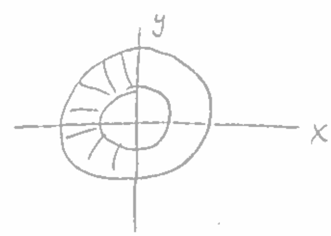
A. $D = \{\ (x, y) \mid 1 \le x^2 + y^2 \le 2,\ y \ge 0\}$

B. $D = \{\ (x, y) \mid 1 \le x^2 + y^2 \le 2,\ y \ge 0,\ x \le 0\}$

C. $D = \{\ (x, y) \mid 1 \le x^2 + y^2 \le 4,\ y \le 0,\ x \le 0\}$

D. $D = \{\ (x, y) \mid 1 \le x^2 + y^2 \le 4,\ x \le 0\}$

E. $D = \{\ (x, y) \mid -\sqrt{4 - y^2} \le x \le +\sqrt{4 - y^2},\ -2 \le y \le 2\}$



(5 pts) 10. Let D be the solid bound above by the sphere $x^2 + y^2 + z^2 = 9$ and below by $z = \sqrt{x^2+y^2}$ with $x \leq 0$; if the function $g(x, y, z) = xz$ is integrated over D using spherical coordinates, the correct integral is:

$xz = \rho\sin\phi\cos\theta \cdot \rho\cos\phi$
$= \rho^2\sin\phi\cos\phi\cos\theta \rightarrow \rho^4\sin^2\phi\cos\phi\cos\theta$

A. $\int_0^{\pi}\int_0^{\pi/4}\int_0^{9} \rho^3\cos\theta\sin\phi\cos\phi \, d\rho \, d\phi \, d\theta$

B. $\int_{\pi/2}^{3\pi/2}\int_0^{\pi/2}\int_0^{9} \rho^3\cos\theta\sin^2\phi\cos\phi \, d\rho \, d\phi \, d\theta$

C. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} \rho^4\cos\theta\sin^2\phi\cos\phi \, d\rho \, d\phi \, d\theta$

D. $\int_{\pi/2}^{3\pi/2}\int_0^{\pi/4}\int_0^{3} \rho^4\cos\theta\sin\phi\cos\phi \, d\rho \, d\phi \, d\theta$

E. none of the above

$\phi = \frac{\pi}{4}$ $\quad \rho^2 = 9 \Rightarrow \rho = 3$

$\int_{\pi/2}^{3\pi/2}\int_0^{\pi/4}\int_0^{3} \rho^4\cos\theta\sin^2\phi\cos\phi \, d\rho \, d\phi \, d\theta$

$x \leq 0 \Rightarrow \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

(5 pts) 11. Let D be the solid bounded above by the sphere $x^2 + y^2 + z^2 = 9$ and below by $z = \sqrt{x^2+y^2}$; if the function $g(x, y, z) = 2xy$ is integrated over D using cylindrical coordinates, the correct integral is:

$2(r\cos\theta)(r\sin\theta)r = 2r^3\sin\theta\cos\theta$

$r \leq z \leq \sqrt{9-r^2}$

A. $\int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_r^{\sqrt{3-r^2}} 2r^2\cos\theta\sin\theta \, dz \, dr \, d\theta$

B. $\int_0^{2\pi}\int_0^{3/\sqrt{2}}\int_r^{\sqrt{9-r^2}} 2r^3\cos\theta\sin\theta \, dz \, dr \, d\theta$

C. $\int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_r^{\sqrt{9-r^2}} 2r^3\cos\theta\sin\theta \, dz \, dr \, d\theta$

D. $\int_0^{2\pi}\int_0^{3\sqrt{2}/2}\int_r^{\sqrt{9-r^2}} 2r^2\cos\theta\sin\theta \, dz \, dr \, d\theta$

E. none of the above

$x^2+y^2+x^2+y^2 = 9$
$x^2+y^2 = \frac{9}{2}$ $\quad r = \frac{3}{\sqrt{2}}$

$R \rightarrow D$

(5 pts) 12. Let $T:\ (u, v, w) \rightarrow (x, y, z)$ be a differentiable, one-to-one transformation which maps a solid R to a solid D; how many of the following are true?

i. The Jacobian of the transformation is calculated using a determinant. ✓

ii. The Jacobian of the transformation is a matrix-valued function. ✗

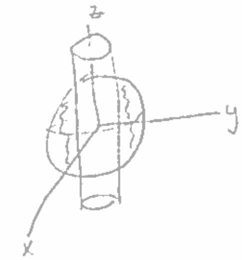
iii. The Jacobian of the transformation is a scalar-valued function. ✓

iv. The volume of D is equal to $\int\int\int_R 1 \, du \, dv \, dw$. ✗

$|J(u,v,w)|$

A. 0 $\quad$ B. 1 $\quad$ C. 2 $\quad$ D. 3 $\quad$ E. 4

(5 pts) 13. Let D be the solid which lies inside the sphere $x^2 + y^2 + z^2 = 25$ and outside of the cylinder $x^2 + y^2 = 4$; the volume of this solid is given by:



A. $\int_0^{2\pi} \int_2^5 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta$

B. $\int_0^{2\pi} \int_0^{\pi} \int_2^5 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

C. $4 \int_2^5 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} 1\, dz\, dy\, dx$ ×

D. $\int_0^{2\pi} \int_0^{\pi} \int_1^4 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

E. $\int_0^{2\pi} \int_4^{25} \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta$ ×

(5 pts) 14. Bonus. The density function of a thin circular plate of radius R is given by $\rho = \sqrt{x^2 + y^2}$; what is the mass of this plate?

A. πR^2

B. $3\pi R^3/2$

C. $2\pi R^3/3$

D. $\pi R^3/3$

E. $2\pi R^2$

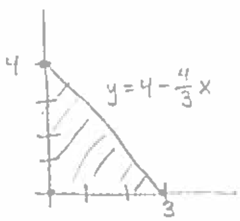
$$M = \iint_R \rho(r,\theta)\, dA = \int_0^{2\pi} \int_0^R r^2\, dr\, d\theta = 2\pi \int_0^R r^2\, dr$$

$$= 2\pi \left(\frac{r^3}{3}\Big|_0^R \right)$$

$$= \frac{2\pi R^3}{3}$$

Name: ______________________ UF-ID: ________________ Section: ____________

(7 pts) 1. Set up but do not evaluate the integral to determine the average value of $f(x,y) = x - y^2$ on the triangle with vertices $(0,0)$, $(0,4)$, and $(3,0)$.



$A(D) = \frac{1}{2} \cdot 4 \cdot 3 = 6$

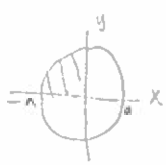
$$\bar{f} = \frac{1}{A(D)} \iint_D f(x,y)\,dA$$

$$= \boxed{\frac{1}{6} \int_0^3 \int_0^{4-\frac{4}{3}x} (x - y^2)\,dy\,dx}$$

(10 pts) 2. Transform the integral $\int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_0^a \rho^4 \sin^2\phi \sin\theta\, d\rho\, d\phi\, d\theta$ from spherical to Cartesian coordinates. Do not evaluate.

$\rho^2 \sin\phi \sin\theta\, (\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta)$

$y(x^2+y^2+z^2)^{1/2}\, dV$

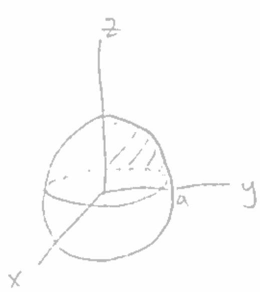


$\rho = a \Rightarrow x^2+y^2+z^2 = a^2$

$z = \pm\sqrt{a^2-x^2-y^2}$

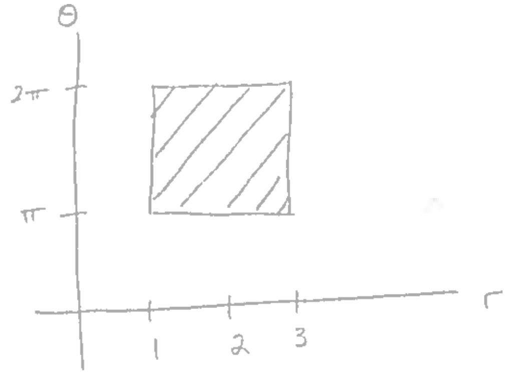
$0 \le \phi \le \frac{\pi}{2} \Rightarrow z \ge 0$

$$\boxed{\int_{-a}^{0} \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} y(x^2+y^2+z^2)^{1/2}\, dz\, dy\, dx}$$

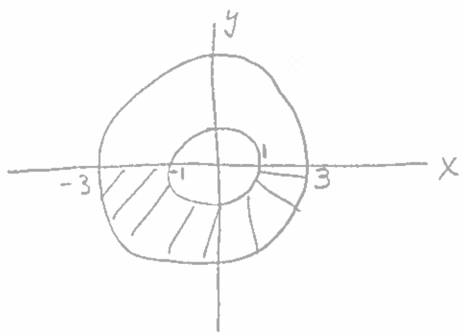


(11 pts) 3. Let R be a region in the r, θ-plane defined by $R = \{\ (r, \theta) \mid 1 \leq r \leq 3,\ \pi \leq \theta \leq 2\pi\ \}$.

(2 pts) a. Sketch this region in the r, θ-plane.



(4 pts) b. Sketch the region D in the x, y-plane which is the image of R under the standard polar transformation.



(2 pts) c. Calculate the Jacobian $J(r, \theta)$ for the standard polar transformation.

$x = r\cos\theta,\ y = r\sin\theta$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

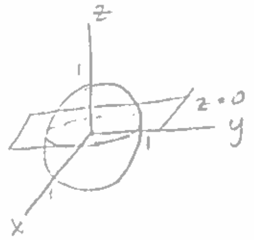
(3 pts) d. Express the integral $\int\int_D x/(x^2 + y^2)\, dA$ as an iterated integral in the r, θ coordinate system (do not evaluate the integral).

$x = r\cos\theta$, $r\,dr\,d\theta$ $\quad \frac{\cancel{r^2}\cos\theta}{\cancel{r^2}} = \cos\theta\, dr\, d\theta$

$x^2 + y^2 = r^2$

$$\boxed{\int_{\pi}^{2\pi} \int_{1}^{3} \cos\theta\, dr\, d\theta}$$

(7 pts) 4. Find the average value of the function $h(x,y,z) = \sqrt{\pi}$ over the solid bounded above by the surface $x^2 + y^2 + z^2 = 1$ and below by $z = 0$. Make sure you show all of your work.



$\rho^2 = 1 \Rightarrow \rho = 1 \qquad z \geq 0$

$V = \frac{\frac{4}{3}\pi r^3}{2} = \frac{2}{3}\pi$

$\bar{h} = \frac{1}{V(W)} \iiint_W h \, dV$

$= \frac{3}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \sqrt{\pi}\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= \frac{3\sqrt{\pi}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^3}{3}\Big|_0^1 \sin\phi \, d\phi \, d\theta$

$= \frac{3}{2\sqrt{\pi}} \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{3} \sin\phi \, d\phi \, d\theta$

$= \frac{1}{2\sqrt{\pi}} \int_0^{2\pi} -\cos\phi \Big|_0^{\pi/2} d\theta$

$= \frac{1}{2\sqrt{\pi}} \int_0^{2\pi} (0+1) \, d\theta$

$= \frac{1}{2\sqrt{\pi}} \int_0^{2\pi} d\theta = \frac{1}{2\sqrt{\pi}} \theta\Big|_0^{2\pi} = \frac{2\pi}{2\sqrt{\pi}} = \sqrt{\pi}$