Free Fall and Air Resistance

In the first part of this project, we add the influence of air resistance to the model of free fall. Denote by \( x(t) \) the distance the object has fallen after \( t \) seconds. Note that we are reversing the orientation from the model given in Section 1.1, where we looked at height rather than distance; consequently, gravity is now counted positively rather than negatively since it increases the distance traveled. The force due to gravity is given by

\[
F_g = mg,
\]

where \( m \) is the mass of the object and \( g \) is the gravitational acceleration constant. We assume that the air resistance is proportional to the velocity of the object, giving the equation

\[
F_a = -bv(t),
\]

where \( b > 0 \) is the proportionality constant, and the negative sign indicates that this force is opposing the direction of motion. Therefore, using the fact that \( a = \frac{dv}{dt} \) and Newton’s second law, we have the DE

\[
md\frac{dv}{dt} = mg - bv.
\]

If we are given the initial velocity \( v_0 \), we can solve the resulting IVP.

**Problem 1.** (2 points) Solve the DE (1) subject to the initial condition \( v(0) = v_0 \) by using an integrating factor.

By taking the vertical position at \( t = 0 \) to be \( x = 0 \), we can now find the distance function \( x(t) \).

**Problem 2.** (2 points) Solve the IVP \( x'(t) = v(t), \; x(0) = 0 \) by integrating your answer to Problem 1. Then approximate the distance traveled by an object of mass 2 kg dropped with initial velocity of 3 m/sec after 4 seconds, assuming that the air resistance constant is \( b = 1 \). Use the approximation \( g = 9.8 \text{ m/sec}^2 \) and round to the nearest tenth.

Logistic Population Model

The simplest population model is that of exponential growth, namely \( p(t) = p_0e^{kt} \) where \( p_0 \) is the initial population and \( k \) is a proportionality constant. However, this model only accounts for death by natural causes. In either a predator-prey environment or a human population, we must also account for deaths due to malnutrition or inadequate resources, diseases, crimes, etc. Doing this leads to the following logistic model:

\[
\frac{dp}{dt} = -Ap(p - d), \quad p(0) = p_0,
\]

where \( A \) and \( d \) are constants determined by the particular system being investigated.
Problem 3. (2 points) Assuming that $A = 1/10$ and $d = 20$, solve the separable equation (2) for $p(t)$ and use your answer to estimate the population of the system after 2 years, assuming the initial population is 10. Round to two decimal places. (Hint: You will need to use a partial fraction decomposition.)

Electrical Circuits

We consider an $RL$ circuit, consisting of a resistor with resistance $R$, an inductor with inductance $L$, and a voltage source $E(t)$. One of Kirchhoff’s laws states that the sum of the voltage drops around any closed loop is zero. If we let $E_R$ denote the voltage drop across the resistor and $E_L$ the voltage drop across the inductor, we get the equation

$$(3) \quad E(t) - E_R - E_L = 0 \quad \Rightarrow \quad E_R + E_L = E(t).$$

The sign of the voltage source is positive because it adds voltage to the system. Ohm’s law tells us that $E_R = RI$, where $I$ is the current passing through the resistor, and using Faraday’s law and Lenz’s law one can show that $E_L = L\frac{dI}{dt}$. Therefore, we obtain the DE

$$(4) \quad L\frac{dI}{dt} + RI = E(t).$$

Problem 4. (2 points) For an $RL$ circuit with a 3 $\Omega$ resistor and a 1 $H$ inductor driven by a voltage $E(t) = \cos(3t) \ V$, use an integrating factor to determine the equation for the current through the circuit by solving the IVP given by equation (4) and the initial condition $I(0) = 1 \ A$. (Hint: the integration will involve a “wraparound” integration by parts.)