Project B: Examining Mass-Spring Systems

Worth 9 points, due on Friday, July 22

Free Motion

Recall from Section 4.1 that an equation governing the motion of a mass-spring system is

$$my'' + by' + ky = F_{ext}(t), \tag{1}$$

where m is the mass, b is the damping coefficient due to the force of friction, k is the stiffness of the spring, and $F_{ext}(t)$ represents the influence of forces external to the system. In the first part of this project, we assume there are no external forces, so that the equation (1) is homogeneous.

The simplest scenario is when there is no damping (friction) present; then $b = 0$ and we have

$$my'' + ky = 0. \tag{2}$$

Problem 1. (2.5 points) Solve the DE (2) subject to the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ to get the equation of motion for any free, undamped mass-spring system. Let ω represent the expression $\sqrt{k/m}$. (Your answer should be a function of t which involves the unspecified constants y_0, v_0, ω.) Use the result to find the equation of motion for a system with mass $m = 1$ and spring constant $k = 64$ released from rest 1 unit to the right of the equilibrium position. Sketch the graph of your equation.

In many systems, we want to minimize the oscillatory motion of solutions (think shock absorbers in your car, bridges, etc.). Solutions tend to zero most quickly when the system is *critically damped*; this occurs when the discriminant $b^2 - 4mk$ of the auxiliary equation for (1) equals zero.

Problem 2. (2.5 points) For the system described in Problem 1 (that is, with mass $m = 1$ and spring constant $k = 64$), determine the value of the damping coefficient b so that the system is critically damped. Then substitute the known values m, b, and k into (1) with $F_{ext}(t) = 0$ and solve the resulting DE to determine the equation of motion for the critically damped system, assuming again that the mass is released from rest 1 unit to the right of the equilibrium position. Sketch the graph of your equation.

Forced Motion

We now consider the case where the mass-spring system is being acted on by a periodic external force. We shall ignore friction and take the damping coefficient $b = 0$. This leads to the DE

$$my'' + ky = F\cos(\lambda t), \tag{3}$$

where F is the amplitude of the external force and λ is the forcing frequency. We assume first that $\lambda \neq \omega = \sqrt{k/m}$.

Problem 3. (2 points) Use the method of undetermined coefficients and the homogeneous solution from the first part of Problem 1 to solve the DE (3) subject to the initial conditions $y(0) = 0$ and $y'(0) = 0$. (Your answer should be a function of t which involves the constants F, m, k, λ, ω.)

Now we examine the case where the forcing frequency λ equals the natural frequency ω, which results in the phenomenon of *resonance*:

$$my'' + ky = F\cos(\omega t) \tag{4}$$

Problem 4. (2 points) Use the method of undetermined coefficients and the homogeneous solution from the first part of Problem 1 to find the general solution to the DE (4). (Hint: use the fact that $\omega^2 = k/m$ to cancel out terms after substituting derivatives into the equation.) What happens to the amplitude of the oscillations as $t \to \infty$?