Project C Solutions

(1) $y'' = -4e(t) \qquad y(0)=0,\ y'(0)=0 \qquad$ where $\quad e(t) = y - g(t) \doteq y - 3t.$

$$\mathcal{L}\{y''\} = -4\,\mathcal{L}\{e\}$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

$$\mathcal{L}\{e\} = \mathcal{L}\{y-3t\} = Y(s) - 3\mathcal{L}\{t\} = Y(s) - \frac{3}{s^2}$$

Substituting gives $\quad s^2Y(s) = -4\left(Y(s) - \frac{3}{s^2}\right) = -4Y(s) + \frac{12}{s^2}$

$$\Rightarrow \quad (s^2+4)Y(s) = \frac{12}{s^2}$$

$$\Rightarrow \quad \underline{Y(s) = \frac{12}{s^2(s^2+4)}}$$

$$\frac{12}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+2D}{s^2+4} \quad \Rightarrow \quad 12 = As(s^2+4) + B(s^2+4) + (Cs+2D)s^2$$

Take $s=0$: $\quad 12 = 4B \Rightarrow \underline{B=3}$

Take $s=1$: $\quad 12 = 5A + 5B + C + 2D \quad \Rightarrow \quad 5A + C + 2D = -3 \qquad 5A + C = 0 \Rightarrow C = -5A$

$s=-1$: $\quad 12 = -5A + 5B - C + 2D \qquad -5A - C + 2D = -3 \qquad \Rightarrow \underline{C=0}$

$4D = -6$

$\underline{D = -\frac{3}{2}}$

$s=2$: $\quad 12 = 16A + 8(3) + 4(-10A-3)$

$12 = -24A + 12 \Rightarrow -24A = 0 \Rightarrow \underline{A=0}$

[OR matching coefficients gives

$$(A+C)s^3 + (B+2D)s^2 + 4As + 4B = 12 \quad \Rightarrow \quad 4A = 0 \Rightarrow \underline{A=0}$$

$A + C = 0 \Rightarrow \underline{C=0}$

$B + 2D = 0 \Rightarrow \underline{D} = -B/2 = \underline{-3/2}$

$4B = 12 \Rightarrow \underline{B=3}$]

So $\quad \underline{Y(s) = \frac{3}{s^2} - \frac{3}{2}\cdot\frac{2}{s^2+4}} \quad \Rightarrow \quad y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$

$$\boxed{y(t) = 3t - \frac{3}{2}\sin(2t)}$$

Thus, $\quad e(t) = y(t) - 3t = 3t - \frac{3}{2}\sin(2t) - 3t = \boxed{-\frac{3}{2}\sin(2t)}.$

Circuits

(4) $50I_1' + 80I_2 = 160$

(5) $50I_1' + 800q_3 = 160$

(6) $I_1 = I_2 + I_3$

$q_3' = I_3$

$q_3(0) = 0.5$

$I_3(0) = 0$

With $x_1 = I_1$, $x_2 = I_1'$, $x_3 = I_2$:

(7) $x_1' = x_2$

(8) $x_2' = -16x_1 + 16x_3$

(9) $x_3' = 10x_1 - 10x_3$

(2) $\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ -16 & 0 & 16 \\ 10 & 0 & -10 \end{bmatrix} \vec{x}$

Char. eqn.: $\begin{vmatrix} -r & 1 & 0 \\ -16 & -r & 16 \\ 10 & 0 & -10-r \end{vmatrix} = -r(r^2+10r) - (16r+160-160)$

$= -r(r^2+10r+16)$

$= -r(r+2)(r+8)$

$\Rightarrow$ Eigenvalues are $r=0$, $r=-2$, $r=-8$.

$r=0$: $\begin{bmatrix} 0 & 1 & 0 \\ -16 & 0 & 16 \\ 10 & 0 & -10 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \begin{matrix} x_2 = 0 \\ x_1 = x_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector

$r=-2$: $\begin{bmatrix} 2 & 1 & 0 \\ -16 & 2 & 16 \\ 10 & 0 & -8 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \begin{matrix} x_2 = -2x_1 \\ x_3 = \frac{5}{4}x_1 \end{matrix} \Rightarrow \begin{bmatrix} 4 \\ -8 \\ 5 \end{bmatrix}$ is an eigenvector

$r=-8$: $\begin{bmatrix} 8 & 1 & 0 \\ -16 & 8 & 16 \\ 10 & 0 & -2 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \begin{matrix} x_2 = -8x_1 \\ x_3 = 5x_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 \\ -8 \\ 5 \end{bmatrix}$ is an eigenvector

Gen. soln: $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2e^{-2t}\begin{bmatrix} 4 \\ -8 \\ 5 \end{bmatrix} + c_3e^{-8t}\begin{bmatrix} 1 \\ -8 \\ 5 \end{bmatrix}}$

$x_1 = I_1 = c_1 + 4c_2e^{-2t} + c_3e^{-8t}$

$x_3 = I_2 = c_1 + 5c_2e^{-2t} + 5c_3e^{-8t}$

By (6), $I_3 = I_1 - I_2 = -c_2e^{-2t} - 4c_3e^{-8t}$.

(3) (a) By (5), $50I_1'(0) + 800q_3(0) = 160 \Rightarrow 50I_1'(0) = 160 - 400 = -240 \Rightarrow \underline{I_1'(0) = -\frac{24}{5}}$

From #2, $x_2 = -8c_2e^{-2t} - 8c_3e^{-8t} \Rightarrow x_2(0) = -8c_2 - 8c_3 = -\frac{24}{5} \Rightarrow c_2 + c_3 = \frac{3}{5}$

$\Rightarrow c_3 = \frac{3}{5} - c_2$

(b) $I_3(0) = -c_2 - 4c_3 = 0 \Rightarrow -c_2 - 4(\frac{3}{5} - c_2) = 0 \Rightarrow 3c_2 = \frac{12}{5} \Rightarrow \underline{c_2 = \frac{4}{5}}$

So $c_3 = \frac{3}{5} - \frac{4}{5} = \underline{-\frac{1}{5}}$

(c) By (4), $50I_1'(0) + 80I_2(0) = 160 \Rightarrow 80I_2(0) = 160 + 240 = 400 \Rightarrow \underline{I_2(0) = 5}$

From #2, $I_2(0) = c_1 + 5c_2 + 5c_3 = 5 \Rightarrow c_1 + 5(\frac{4}{5}) + 5(-\frac{1}{5}) = c_1 + 4 - 1 = 5 \Rightarrow \underline{c_1 = 2}$

(d) $\boxed{I_1 = 2 + \frac{16}{5}e^{-2t} - \frac{1}{5}e^{-8t}}$, $\boxed{I_2 = 2 + 4e^{-2t} - e^{-8t}}$, $\boxed{I_3 = -\frac{4}{5}e^{-2t} + \frac{4}{5}e^{-8t}}$