Project C: Control Theory and Circuits

Worth 8 points, due on Friday, August 5

Control Theory

Consider a mechanical system that models an automatic pilot. The system applies torque to the steering control shaft so that the vessel (a boat or plane, for example) will follow the desired course. Let $y(t)$ be the true direction of the vessel at time t and $g(t)$ be the desired direction at time t. Then the error is given by the function

$$e(t) = y(t) - g(t). \tag{1}$$

Suppose that our system measures the error and provides feedback to the steering shaft in the form of torque proportional to the error but opposite in sign (an attempt to "course correct"); then Newton's second law in terms of torque gives the DE

$$Iy''(t) = -ke(t), \tag{2}$$

where I is the moment of inertia of the steering shaft, y'' is the angular acceleration, and $k > 0$ is a proportionality constant.

We'll consider a system where the steering shaft has moment of inertia $I = 1$ and the proportionality constant is $k = 4$. If the steering shaft is initially at rest in the zero direction, this produces the IVP

$$y'' = -4e(t), \quad y(0) = 0, \quad y'(0) = 0. \tag{3}$$

Problem 1. (3 points) Use the technique of Laplace transforms to solve the IVP (3) for the true course $y(t)$ assuming that the desired course has equation $g(t) = 3t$. Use the result to find the equation of the error function $e(t)$.

- You may wish to use the notation $Y(s) = \mathscr{L}\{y\}(s), E(s) = \mathscr{L}\{e\}(s)$.
- Hint 1: Use equation (1) to find a formula for $E(s)$ in terms of $Y(s)$.
- Hint 2: You may find it easier to equate coefficients in the partial fraction decomposition rather than making substitutions.

Circuits

In this section, we consider a type of electrical circuit called an RLC network, which consists of a resistor, inductor, and capacitor connected either in series or parallel, in addition to a voltage source. Our particular circuit is a series LC circuit with a $50H$ inductor, an 80Ω resistor and $\frac{1}{800}F$ capacitor in parallel, and a $160V$ voltage source. There are three currents in the circuit, governed by the following equations:

$$50I_1'(t) + 80I_2(t) = 160, \tag{4}$$

$$50I_1'(t) + 800q_3(t) = 160, \tag{5}$$

$$I_1(t) = I_2(t) + I_3(t), \tag{6}$$

where $q_3(t)$ is the charge on the capacitor, $q_3'(t) = I_3(t)$ and initial conditions are $q_3(0) = 0.5$ coulombs and $I_3(0) = 0$ amps.

We can manipulate these equations to derive the following linear homogeneous system in normal form, where $x_1 = I_1, x_2 = I_1'$, and $x_3 = I_2$:

$$x_1' = x_2, \tag{7}$$

$$x_2' = -16x_1 + 16x_3, \tag{8}$$

$$x_3' = 10x_1 - 10x_3. \tag{9}$$

Problem 2. (3 points) Write the above system of equations (7)-(9) as a matrix equation, and use matrix methods to find the general solution to the system. Use your answer to write equations for the currents I_1, I_2, I_3 in terms of undetermined constants.

Problem 3. (2 points) Determine the constants in the answer to Problem 2 by following these steps:

(a) Substitute the IC for q_3 into equation (5) to solve for $I_1'(0) = x_2(0)$, and then use this initial condition with the equation for x_2 from Problem 2 to write one of the constants in terms of a second;

(b) Use the IC for I_3 and the substitution you found in part (a) in the expression for I_3 from Problem 2 to solve for the two constants involved in that expression;

(c) Plug the value you found for $I_1'(0)$ in part (a) into equation (4) to solve for $I_2(0)$, and use this IC with the expression for I_2 from Problem 2 to determine the final constant.

(d) Now that you have found all the constants, write the equations for the currents I_1, I_2, I_3.