HOW TO UNDERSTAND QUADRIC SURFACES

One approach to classifying quadric surfaces is simply memorizing the general equations that correspond to each surface. While this is possible, it is difficult and gives no geometric intuition about the shapes of these surfaces. It is much easier to understand these surfaces by examining their traces.

Definition 1. The trace of a three-dimensional surface is the intersection of that surface with a plane that is parallel to one of the coordinate planes.

In particular, these planes will have an equation of the form $x = C, y = C$, or $z = C$ for some constant C. Planes with equations of the form $x = C$ or $y = C$ are called vertical traces because they are vertical; planes with equations of the form $z = C$ are called horizontal traces because they lie flat. When the constant $C = 0$ we have specific names for the traces: $z = 0$ corresponds to the x, y-trace, $y = 0$ corresponds to the x, z-trace, and $x = 0$ corresponds to the y, z-trace. To find a trace, simply substitute the value given by the plane into the equation of the surface to get an equation in two variables. (This is probably all you will have to do on the exam.) The result will always be a 2-dimensional shape (or possibly empty/nonexistent); the type of shapes obtained from an equation give information by which the surface can be classified.

There are four main classes of quadric surfaces, some with subclasses. Below we describe the traces of each class.

The simplest class is the cylinders; a cylinder is given by any equation that involves only two of the variables x, y, and z and leaves the third free to be any value whatsoever. Cylinders are described further with an adjective which identifies the shape of the cross-sections of the cylinder. Perhaps the most familiar example is $x^2 + y^2 = 1$: the x, y-trace of this surface (substituting $z = 0$) is the unit circle in the x, y-plane. Indeed, intersecting with the plane $z = C$ for any constant C will give the same circle, so this surface is a circular cylinder. Likewise, the equation $z = x^2 + 4$ represents a parabolic cylinder (the x, z-trace is a parabola and y is free to take on any value) and the equation $y^2 - z^2 = 1$ represents a hyperbolic cylinder (the y, z-trace is a hyperbola and x is free).

The next easiest class to understand is the ellipsoids. Every trace of an ellipsoid is an ellipse; a special case of this is a sphere, where every trace is actually a circle. So ellipsoids are just spheres that have been "stretched out". The general equation looks like this:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

The third class contains the paraboloids. There are two varieties: elliptical paraboloids and hyperbolic paraboloids. The vertical traces of all paraboloids are parabolas. If the horizontal trace is an ellipse, you have an elliptic paraboloid; if the horizontal trace is a hyperbola, then (you guessed it!) it's a hyperbolic paraboloid. The general equation looks like this:

$$z = \left(\frac{x}{a}\right)^2 \pm \left(\frac{y}{b}\right)^2,$$

where "+" yields an elliptic paraboloid and "-" yields a hyperbolic paraboloid.

Probably the most difficult class is the hyperboloids, which come in three varieties: the hyperboloid of one sheet, the hyperboloid of two sheets, and the cone. For all three the vertical traces will be hyperbolas and the horizontal traces will be ellipses (or empty). To distinguish the three varieties, we will consider the x, y-trace of the general equation:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + C,$$

where $C = 1, -1$, or 0.

If $C = 1$, then the x, y-trace of the equation is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, which is an ellipse; this indicates the surface is a hyperboloid of one sheet.

If $C = -1$, then the x, y-trace of the equation is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = -1$. Note that the left-hand side must be nonnegative, but the right-hand side is negative; therefore, this equation has no solution and the trace is empty. This indicates the surface is a hyperboloid of two sheets.

If $C = 0$, then the x, y-trace of the equation is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 0$. The only solution to this equation is when x and y are both 0, so the trace is the origin, a single point. This indicates the surface is a cone. You can think of a cone as a hyperboloid of one sheet where the middle has been "pinched" to a single point.

There you have it - those are all the quadric surfaces, and no memorization of formulas! The traces themselves will tell you what the surface is.