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Quiz 5

You must give complete, mathematically correct proofs to receive full credit!!

Problem 1. (5 points) How many elements of order 9 does $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ have? Justify your answer.

The element $(\bar{a}, \bar{b}) \in \mathbb{Z}_3 \oplus \mathbb{Z}_9$ has order 9 if $\operatorname{lcm}(|\bar{a}|, |\bar{b}|) = 9$. By Lagrange's Thrm, $|\bar{a}| \in \{1, 3\}$ and $|\bar{b}| \in \{1, 3, 9\}$. There are two cases which give $\operatorname{lcm}(|\bar{a}|, |\bar{b}|) = 9$:

Case 1: $|\bar{a}| = 1$, $|\bar{b}| = 9$. Then $\bar{a} = \bar{0} \in \mathbb{Z}_3$, and there are $\phi(9) = 6$ elements of order 9 in $\mathbb{Z}_9$, so this case yields $1 \cdot 6 = 6$ elements.

Case 2: $|\bar{a}| = 3$, $|\bar{b}| = 9$. There are still 6 choices for $\bar{b}$, but now there are $\phi(3) = 2$ choices for $\bar{a}$, so this case yields $2 \cdot 6 = 12$ elements.

Thus, $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ has 18 elements of order 9.

Problem 2. (5 points) Is $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \cong \mathbb{Z}_{60} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$? Justify your answer.

Using the fact that $\mathbb{Z}_m \oplus \mathbb{Z}_n = \mathbb{Z}_{mn}$ iff $\gcd(m,n) = 1$ and rearranging the factors as needed, we have

$$\begin{aligned}
\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 &\cong (\mathbb{Z}_2 \oplus \mathbb{Z}_5) \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_4) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \\
&\cong (\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \oplus \mathbb{Z}_2 \\
&\cong \mathbb{Z}_{60} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2.
\end{aligned}$$

So yes, they are isomorphic.