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Quiz 6

You must give complete, mathematically correct proofs to receive full credit!!

Problem 1. (5 points) Let G be a nonabelian group of order p^3 for some prime p, and suppose that $\mathbf{Z}(G)$ is nontrivial. Prove that $|\mathbf{Z}(G)| = p$.

Proof By Lagrange's Thrm, $|Z(G)| \in \{1, p, p^2, p^3\}$. Since $Z(G)$ is nontrivial, $|Z(G)| \neq 1$, and since G is nonabelian, $Z(G) \neq G \Rightarrow |Z(G)| \neq p^3$. Suppose that $|Z(G)| = p^2$. Then $|G/Z(G)| = \frac{p^3}{p^2} = p$ and by a Corollary of Lagrange's Thrm, $G/Z(G)$ is cyclic. Thus, G is abelian by the G/Z Thrm, but this is a contradiction. Therefore, $|Z(G)| = p$. ▨

Problem 2. (5 points) Let $\phi : G \to H$ be a group homomorphism and let $N \trianglelefteq H$. Prove that $\phi^{-1}(N) \trianglelefteq G$.

Proof Let $y \in \phi^{-1}(N)$ and $x \in G$; we WTS that $xyx^{-1} \in \phi^{-1}(N)$. Now $\exists n \in N$ s.t. $\phi(y) = n$. We then have

$$\phi(xyx^{-1}) = \phi(x)\phi(y)\phi(x^{-1}) = \phi(x)\, n\, \phi(x)^{-1} \in N \quad \text{since } N \trianglelefteq H.$$

Thus, $xyx^{-1} \in \phi^{-1}(N)$, so $\phi^{-1}(N) \trianglelefteq G$ by the Normal Subgroup Test. ▨