Lecture 15 WA (Sec. 14.7)

(5) 1. $f(x,y) = x^2 + xy + y^2 + 2y$

$f_x = 2x + y = 0 \Rightarrow y = -2x \Rightarrow y = -\frac{4}{3}$

$f_y = x + 2y + 2 \Rightarrow x - 4x + 2 = 0 \Rightarrow 3x = 2 \Rightarrow x = \frac{2}{3}$

$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 1$, so $D = 4 - 1 = 3 > 0$, $f_{xx} > 0 \Rightarrow$

$\boxed{(\frac{2}{3}, -\frac{4}{3}) \text{ is a local min}}$ $\quad f(\frac{2}{3}, -\frac{4}{3}) = \frac{4}{9} - \frac{8}{9} + \frac{16}{9} - \frac{8}{3} = -\frac{4}{3}$

(6) 2. $f(x,y) = xy - 5x - 5y - x^2 - y^2$

$f_x = y - 5 - 2x = 0 \Rightarrow y = 2x + 5 \Rightarrow y = -5$

$f_y = x - 5 - 2y = 0 \Rightarrow x - 5 - (4x + 10) = 0 \Rightarrow 3x = -15 \Rightarrow x = -5$

$D = f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3 > 0$, $f_{xx} < 0$, so

$f\boxed{(-5,-5)} = 25 + 25 + 25 - 25 - 25 = 25$ is a $\boxed{\text{local max}}$

(10) 3. $f(x,y) = 3 - x^4 + 2x^2 - y^2$

$f_x = -4x^3 + 4x = 4x(1 - x^2) = 4x(1-x)(1+x) = 0 \Rightarrow x = 0, 1, -1$

$f_y = -2y = 0 \Rightarrow y = 0$; $f_{xx} = -12x^2 + 4$, $f_{yy} = -2$, $f_{xy} = 0$

$D = 24x^2 - 8$. $D(0,0) = -8 < 0 \Rightarrow \boxed{(0,0,3)}$ is a $\boxed{\text{saddle point}}$

$D(1,0) = 16 > 0$, $f_{xx}(1) = -8 < 0 \Rightarrow \boxed{(1,0,4)}$ is a $\boxed{\text{local max}}$

$D(-1,0) = 16 > 0$, $f_{xx}(-1) = -8 < 0 \Rightarrow \boxed{(-1,0,4)}$ is a $\boxed{\text{local max}}$.

(11) 4. $f(x,y) = 3x^3 - 9x + 9xy^2$

$f_x = 9x^2 - 9 + 9y^2$, $x = 0$: $9y^2 - 9 = 0 \Rightarrow y = \pm 1$; $y = 0$: $9x^2 - 9 = 0 \Rightarrow x = \pm 1$.

$f_y = 18xy = 0 \Rightarrow x = 0$ or $y = 0$. CP's: $(0,1), (0,-1), (1,0), (-1,0)$.

$f_{xx} = 18x$, $f_{yy} = 18x$, $f_{xy} = 18y \Rightarrow D(x,y) = 324(x^2 - y^2)$

$D(0,1) = -324 < 0 \Rightarrow \boxed{f(0,1) = 0}$ is a $\boxed{\text{saddle point}}$

$D(0,-1) = -324 < 0 \Rightarrow \boxed{f(0,-1) = 0}$ is a $\boxed{\text{saddle point}}$

$D(1,0) = 324 > 0$, $f_{xx}(1,0) = 18 > 0 \Rightarrow \boxed{f(1,0) = -6 \text{ is a local min}}$

$D(-1,0) = 324 > 0$, $f_{xx}(-1,0) = -18 < 0 \Rightarrow \boxed{f(-1,0) = 6 \text{ is a local max}}$

12) 5. $f(x,y) = x^3 + y^3 - 3x^2 - 6y^2 - 9x$

$f_x = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1) = 0 \Rightarrow x = 3, -1$

$f_y = 3y^2 - 12y = 3y(y-4) = 0 \Rightarrow y = 0, y = 4$ CPs: $(3,0)(3,4), (-1,0), (-1,4)$

$f_{xx} = 6x - 6$, $f_{yy} = 6y - 12$, $f_{xy} = 0 \Rightarrow D(x,y) = 36(x-1)(y-2)$

$D(3,0) < 0 \Rightarrow \boxed{f(3,0) = -27 \text{ is a saddle point}}$

$D(3,4) > 0$, $f_{xx}(3,4) > 0 \Rightarrow \boxed{f(3,4) = -59}$ is a $\boxed{\text{local min}}$

$D(-1,0) > 0$, $f_{xx}(-1,0) < 0 \Rightarrow \boxed{f(-1,0) = 5}$ is a $\boxed{\text{local max}}$

$D(-1,4) < 0 \Rightarrow \boxed{f(-1,4) = -27 \text{ is a saddle point}}$

(24) 6. $f(x,y) = 9(x-y)e^{-x^2-y^2}$

$f_x = 9(e^{-x^2-y^2} - 2x(x-y)e^{-x^2-y^2}) = 0 \Rightarrow 1-2x(x-y) = 0 \Rightarrow 2x^2 - 2xy = 1$

$f_y = 9(-e^{-x^2-y^2} - 2y(x-y)e^{-x^2-y^2}) = 0 \Rightarrow -1-2y(x-y) = 0 \Rightarrow 2y^2 - 2xy = 1$

Subtracting gives $2x^2 - 2y^2 = 0 \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$.

$y = x$: $2x^2 - 2x^2 = 0 = 1$ ⓧ $y = -x$: $2x^2 + 2x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$,

so CPs: $(\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})$, $f_{xx} = -18xe^{-x^2-y^2}(1-2x^2+2xy) + (2y-4x)9e^{-x^2-y^2}$

$= 18e^{-x^2-y^2}(-3x + 2x^3 - 2x^2y + y)$; $f_{yy} = -18y\,e^{-x^2-y^2}(-1-2xy+2y^2) + (4y-2x)\cdot 9e^{-x^2-y^2}$

$= 18e^{-x^2-y^2}(3y + 2xy^2 - 2y^3 - x)$; $f_{xy} = -18ye^{-x^2-y^2}(1-2x^2+2xy) + 18xe^{-x^2-y^2}$

$= 18e^{-x^2-y^2}(x - y + 2x^2y - 2xy^2)$. Then $D(\frac{1}{2}, -\frac{1}{2}) = 18e^{-1/2}(-\frac{3}{2} + \frac{1}{4} + \frac{1}{4} - \frac{1}{2}) = (-27e^{-1/2})$

$[18e^{-1/2}(-\frac{3}{2} + \frac{1}{4} + \frac{1}{4} - \frac{1}{2}) = (-27e^{-1/2})] - [18e^{-1/2}(1 - \frac{1}{4} - \frac{1}{4}) = 9e^{-1/2}]^2 = 27^2e^{-1} - 81e^{-1} =$

$648e^{-1} > 0$, $f_{xx}(\frac{1}{2}, -\frac{1}{2}) = -27e^{-1/2} < 0 \Rightarrow$ $\boxed{f(\frac{1}{2}, -\frac{1}{2}) = 9e^{-1/2} \text{ is local max}}$

$D(-\frac{1}{2}, \frac{1}{2}) = (27e^{-1/2})(27e^{-1/2}) - (-9e^{-1/2})^2 = 648e^{-1} > 0$, $f_{xx}(-\frac{1}{2}, \frac{1}{2}) = 27e^{-1/2} > 0 \Rightarrow$

$\boxed{f(-\frac{1}{2}, \frac{1}{2}) = -9e^{-1/2} \text{ is local min}}$

(33) 7. $f(x,y) = x^2 + y^2 + x^2y + 6$ Absolute max/min on $D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$

$f_x = 2x + 2xy = 2x(1+y) = 0 \Rightarrow x = 0$ or $y = -1$ CPs: $(0,0)$ $(\sqrt{2}, -1)$ $(-\sqrt{2}, -1)$

$f_y = 2y + x^2 = 0$; $x = 0 \Rightarrow y = 0$; $y = -1 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$ not in D

$f(0,0) = 6$; $f(x,1) = 2x^2 + 7 \Rightarrow f'(x,1) = 4x = 0 \Rightarrow x = 0$, $f(0,1) = 7$, $f(1,1) = 9$, $f(-1,1) = 9$

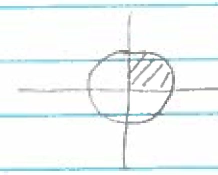
$f(x,-1) = 7$;

$f(1,y) = y^2 + y + 7 \Rightarrow f'(1,y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$, $f(1, -\frac{1}{2}) = 6.75$, $f(1,1) = 9$, $f(1,-1) = 7$

$f(-1,y) = y^2 + y + 7$ "   " $f(-1, -\frac{1}{2}) = 6.75$, $f(-1,1) = 9$, $f(-1,-1) = 7$

Absolute max: 9, absolute min: 6

(36) 8. $f(x,y) = xy^2 + 3$ on $D = \{(x,y) \mid x, y \ge 0,\ x^2 + y^2 \le 3\}$



$f_x = y^2 = 0 \Rightarrow y = 0$

$f_y = 2xy = 0 \Rightarrow x$ is free

L_1: $(y = 0,\ 0 \le x \le \sqrt{3})$ has constant value $f(x,0) = 3$.

L_2: $(x = 0,\ 0 \le y \le \sqrt{3})$ has constant value $f(0,y) = 3$.

L_3: $(y = \sqrt{3 - x^2},\ 0 \le x \le \sqrt{3})$ has $f(x) = x(3 - x^2) + 3 = -x^3 + 3x + 3 \Rightarrow f'(x) = -3x^2 + 3 = 0$

$\Rightarrow -3x^2 = -3 \Rightarrow x = \pm 1$, so $f(0) = 3$, $f(1) = 5$, $f(\sqrt{3}) = 3$

Thus, absolute max is 5, absolute min is 3.

Lecture 15 WA, cont. (2)

(37) 9. $f(x,y) = 2x^3 + y^4 + 3$ on $D = \{(x,y) \mid x^2+y^2 \le 1\}$

$f_x = 6x^2 = 0 \Rightarrow x = 0$; $f_y = 4y^3 = 0 \Rightarrow y = 0$; $f(0,0) = 3$.

Boundary: $y = \sqrt{1-x^2}$ $(-1 \le x \le 1) \Rightarrow f(x, \sqrt{1-x^2}) = 2x^3 + (1-x^2)^2 + 3 =$

$x^4 + 2x^3 - 2x^2 + 4 \Rightarrow f'(x) = 4x^3 + 6x^2 - 4x = 2x(2x^2+3x-2) = 2x(2x-1)(x+2) = 0$

$\Rightarrow x = 0, \frac{1}{2}, -2$. Throw out -2, but $f(0) = 4$, $f(\frac{1}{2}) = \frac{61}{16} = 3.8125$, $f(1) = 5$, $f(-1) = 1$.

$y = -\sqrt{1-x^2}$ $(-1 \le x \le 1) \Rightarrow f(x, -\sqrt{1-x^2}) = 2x^3 + (1-x^2)^2 + 3$, same as before.

So absolute min is 1, absolute max is 5.

(41) 10. Find shortest distance from $(2,0,-5)$ to the plane $x+y+z=5$.

$d = \sqrt{(x-2)^2 + (y-0)^2 + (z+5)^2} \Rightarrow d^2 = (x-2)^2 + y^2 + (10-x-y)^2$

$f_x = 2(x-2) - 2(10-x-y) = 4x + 2y - 24 = 0 \Rightarrow -6y = -16 \Rightarrow y = +\frac{8}{3}$,

$f_y = 2y - 2(10-x-y) = 2x + 4y - 20 = 0$ $\quad x = \frac{14}{3} \Rightarrow \left(\frac{14}{3}, \frac{8}{3}, -\frac{7}{3}\right)$

$\Rightarrow d = \sqrt{(8/3)^2 + (8/3)^2 + (8/3)^2} = \sqrt{\frac{64}{3}} = \boxed{\frac{8}{\sqrt{3}}}$.

(43) 11. Find points on cone $z^2 = x^2+y^2$ closest to $(4,2,0)$.

Minimize $d^2 = (x-4)^2 + (y-2)^2 + (x^2+y^2) = 2x^2 + 2y^2 - 8x - 4y + 20$.

Then $f_x = 4x - 8 = 4(x-2) = 0 \Rightarrow x = 2$, $f_y = 4y - 4 = 4(y-1) = 0 \Rightarrow y = 1$, so

$\boxed{(2, 1, \sqrt{5}),\ (2, 1, -\sqrt{5})}$.

(45) 12. Find three positive numbers with sum 310 & maximum product.

$x + y + z = 310 \Rightarrow z = 310 - x - y$, so $f(x,y) = xy(310-x-y)$. Then

$f_x = y(310-x-y) - xy = 310y - 2xy - y^2 = 0 \Rightarrow y(-y - 2x + 310) = 0 \Rightarrow$ $y = 0$ or $y = -2x + 310$

$f_y = x(310-x-y) - xy = 310x - x^2 - 2xy$

$y = 0$: $x^2 - 310x = 0 \Rightarrow x(x-310) = 0 \Rightarrow x = 0, 310$

$y = -2x + 310$: $310x - x^2 - 2x(-2x+310) = 3x^2 - 310x = x(3x - 310) = 0 \Rightarrow x = 0, \frac{310}{3}$

$f(0,0) = 0$, $f(310,0) = 0$, $f(0,310) = 0$, $f(\frac{310}{3}, \frac{310}{3}) = \left(\frac{310}{3}\right)^3 = \frac{29791000}{27}$

$\boxed{(x,y,z) = \left(\frac{310}{3}, \frac{310}{3}, \frac{310}{3}\right)}$

13. $f(x,y) = 4x + 6y - x^2 - y^2 + 4$ on $D = \{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le 5\}$

$f_x = 4 - 2x = 0 \Rightarrow x = 2$, $f_y = 6 - 2y = 0 \Rightarrow y = 3$, so CP is $(2,3)$

$f(2,3) = 8 + 18 - 4 - 9 + 4 = 17$. Abs. max

L_1: $(x=0)$ $f(0,y) = 6y - y^2 + 4 \Rightarrow f'(y) = -2y + 6 = 0 \Rightarrow y = 3$; $f(0) = 4$, $f(3) = 13$, $f(5) = 9$

L_2: $(y=0)$ $f(x,0) = 4x - x^2 + 4 \Rightarrow f'(x) = -2x + 4 = 0 \Rightarrow x = 2$; $f(0) = 4$, $f(2) = 8$, $f(4) = 4$ Abs. min

L_3: $(x=4)$ $f(4,y) = 6y - y^2 + 4$ [same as above] L_4: $(y=5)$ $f(x,5) = 4x - x^2 + 9$

$f(0) = 9$, $f(2) = 13$, $f(4) = 9$