On the number of groups of order 1024

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Abstract

We report on a recent enumeration of groups of order $2^{10}$ which shows that there are $49487367289$ groups of this order.

Keywords: $p$-groups, Enumeration, $p$-group Generation Algorithm, Small Groups.

2010 Mathematics Subject Classification: 20C40,20B40.

In 1878, Cayley named the problem of finding all the groups of a fixed order as the general problem of groups [1]. Significant progress has been made for small $n$ ($n \leq 20000$) since his call to action [11]. For $p$-groups the difficulties increase rapidly with the exponent. We now outline a brief history of the determination of $2$-groups.

The work of P. Hall, M. Hall, and J. Senior led to a detailed monograph on the determination of the groups of order dividing 64 [2]. The first error free determination of the groups of order 128 appeared in 1990 as a result of the work of Newman, O’Brien, and James [4]. Their work corrected an earlier attempt by Rodemich [3] and was the second of a three paper sequence with the other two papers being written solely by O’Brien. The first of this sequence described the $p$-group generation algorithm algorithm in detail [5], and the last describing its use in the determination of the groups of order 256 [6]. Just over ten years later in 2002, the determination of the groups of order 512, and the enumeration of the groups of order $2^{10}$ were announced in [8]. The number of groups of order $2^{10}$ as reported in 2002 in [8] has since been reported in multiple sources [9], [11] and is available within both the GAP and Magma computer algebra systems, and even appears in Isaacs’ recent group theory textbook [10]. This announced number, however, is incorrect.

In the present work we report on a recent enumeration of the groups of order $2^{10}$ using the same methods as those outlined in [8]. Our enumeration shows that the total number of groups of order $2^{10}$ is actually $49487367289$, not $49487365422$ as earlier reported. Upon discovering this discrepancy, email correspondence with E.A. O’Brien was helpful in identifying the error in the original calculation. The number of immediate descendants of $512\#10493254$ was recorded incorrectly as that value for $512\#10493245$. The number of immediate descendants of $512\#10493254$ was originally recorded as 933 whereas the correct number is 2800. This is the only source of error in the original calculation and accounts for the discrepancy.

Using the algorithms described in [5], [7] we constructed via the $p$-group generation algorithm many of the groups of order $2^{10}$ and completed the enumeration of all the groups of order $2^{10}$ using cohomological techniques. The calculation was performed using GAP4 [14], the GAP Small Groups Library [12], the ANU $p$-Quotient package [13], the EnumPGrp package [15] and a standard parallel process on the HPC cluster at the University of Florida. The HPC cluster consists of 2.3 Ghz Intel Xeon cores and provided at most 200 GB of memory per core during the calculation.

This calculation was performed in two steps. The first step was the following: for every group of order dividing $2^{9}$, apart from the $d$-generator elementary abelian groups for $5 \leq d \leq 9$, we constructed its immediate descendants of order $2^{10}$ using the $p$-group generation algorithm and recorded this number. This portion of the calculation took approximately 2032 hours. The second step was the enumeration of the immediate descendants of order $2^{10}$ of the $d$-generator elementary abelian groups for $5 \leq d \leq 9$ using the cohomological approach of [7]. This part of the computation took less than 1 minute. As these calculations account for all the groups of order $2^{10}$ except the unique elementary abelian group, we have the following result.

**Theorem.** There are precisely $49487367289$ groups of order $2^{10}$.
We record the Frattini quotient rank and exponent-$p$ class of these groups in the following table.

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References


