ADVANCED CALCULUS MAA4102
FINAL EXAM
FALL 2005

Name:

No notes, books, or calculators permitted during the exam.
Each problem is worth 25 points.
Explain all answers!

1.

a. Give a careful statement of the least upper bound principle.

b. Prove: Every bounded increasing sequence converges.

2.

a. Give a careful definition of what it means for a sequence to be *Cauchy*.

b. Prove: If a sequence is Cauchy, then it is bounded.

c. Prove: Every Cauchy sequence converges.

3.

a. Prove: If $f(x) : [a, b] \rightarrow \Re$ is continuous at each $x \in [a, b]$, then $f(x)$ is bounded. (In the proof of this theorem, you are NOT allowed to assume the extremum theorem.)

b. Give a careful statement of the Extremum Theorem.

c. Give a careful statement of Rolle's Theorem.

d. Prove Rolle's Theorem.

4.

a. Give a careful statement of the intermediate value theorem.

b. Prove: If $f(x) : [a, b] \to \Re$ is continuous at each $x \in [a, b]$, $w(x) : [a, b] \to \Re$ is continuous at each $x \in [a, b]$, and $w(x) \geq 0$ for each $x \in [a, b]$, then there is a point $z \in [a, b]$ so that $\int_a^b f(x)w(x)\ dx = f(z) \int_a^b w(x)\ dx$.

5.

a. If $f(x) = x - \frac{1}{2}\sin(x) + 10$, then show that $f(x)$ has at least one real root.

b. If $f(x) = x - \frac{1}{2}\sin(x) + 10$, then show that $f(x)$ cannot have two distinct real roots.

c. If $f(x) : \Re \to \Re$ is differentiable at every $x \in \Re$ and $f(x)$ is strictly increasing, then is it necessarily true that $f'(x) > 0$ for all $x \in \Re$? Explain!

6.

a. Give a careful statement of Taylor's theorem. (Be sure to include the error term.)

b. If $f(x) = \cos(7x)$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $tol = \frac{1}{10^5}$, then find an integer n so that the n^{th} degree Taylor polynomial approximates $f(x)$ with error less than $\frac{1}{10^5}$.

c. Find the Taylor series for the function $f(x) = x\ln(1 - x^3)$.

7.

a. If $f(x) = x - e\ ln(x)$, then show that $f(x)$ is increasing on the interval $[e, +\infty)$.

b. If $f(x) = x - e\ ln(x)$, then show that $f(x) \geq 0$ for all $x \in [e, +\infty)$.

c. Show: $\pi \geq e\ ln(\pi)$.

d. Show: $e^{\pi} \geq \pi^{e}$.

8.

a. Define what it means for the integral to exist. (i.e. Define the symbol $\int_a^b f(x)\ dx$.)

b. If $F(x) = \int_{x^2}^{3x} \sin(x^3)\ dx$, then compute $F'(x)$.

c. Using the $\epsilon - \delta$ definition show that $\lim_{x\to 5} \frac{x^2-25}{x-5} = 10$.