ADVANCED CALCULUS MAA4102

FINAL EXAM

FALL 2006

Name:

No notes, books, or calculators permitted during the exam.
Each problem is worth 25 points.
Explain all answers!

1.

a. Show how the Archimedes/Heron Algorithm can be used to compute the square root of 7. (i. e. If $x_0 = 1$, then give the recursive formula to compute x_{n+1} from x_n).

b. Prove: If $K > 0$, then the Archimedes/Heron Algorithm produces a sequence which is bounded from below by $\sqrt{K}$.

c. Prove: If $K > 0$ and x_n denotes the n^{th} term in the Archimedes/Heron algorithm to approximate $\sqrt{K}$ and $x_0 = 1$, then $x_{n+1} \leq x_n$.

2.

a. Give a careful DEFINITION of what $\lim_{x\to a} f(x) = L$. means.

b. Using the DEFINITION of limit, prove: $\lim_{x\to 3} \frac{x^2-11x+24}{x-3} = -5$.

c. Give a careful DEFINITION of what it means for a function $f(x)$ to be continuous at a point $x = a$.

d. Using the DEFINITION of limit, prove that the function $f(x) = |x|$ is continuous at each $x \in \Re$.

3.

a. Give a careful statement of the Mean Value Theorem.

b. Give a careful proof of the Mean Value Theorem.

c. Prove: If $f(x) : [a, b] \to \Re$ is a differentiable function with the property that $f'(x) = 0$ for all $x \in [a, b]$, then $f(x) = f(a)$ ($= constant$) for all $x \in [a, b]$.

d. Explain the connection between the proposition you just proved in part c (above) and the Fundamental Theorem of Calculus.

4.

a. Give a careful statement of the Intermediate Value Theorem for integrals.

b. Prove the Intermediate Value Theorem for integrals.

5.

a. Give a careful definition of what it means for a series $\sum_{n=0}^{\infty} a_n$ to be convergent.

b. Give a careful statement of the n^{th} root test for a series $\sum_{n=0}^{\infty} a_n$.

c. Prove the n^{th} root test for a series $\sum_{n=0}^{\infty} a_n$. (i.e. Compare this series with another well-known series.)

d. Draw a picture which explains why an alternating series converges.

6.

a. Give a careful statement of the Lagrange Theorem for polynomial interpolation. (Be sure to include the error term.)

b. Given the data $(1,2),(5,47)$, and $(-7,2)$, find a 2-degree polynomial $p_2(x)$ with the property that $p_2(1)=2, p_2(5)=47, p_2(-7)=2$.

c. If $f(x)=e^{3x}$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $tol = \frac{1}{10^5}$, then find an integer n so that the n^{th} degree interpolating polynomial $p_n(x)$ for equally spaced points has the property that $|f(x)-p_n(x)| < \frac{1}{10^5}$, for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

7.

a. If $f(x); [x_0, +\infty) \to \Re$ is a differentiable function and $f''(x) > 0$ for all $x \in [x_0, +\infty)$ and $F(x) = \frac{f(x)-f(x_0)}{x-x_0}$ for all $x > x_0$, then show that $F(x)$ is increasing on the interval $(x_0, +\infty)$.

b. Give a careful DEFINITION of what it means for a function to be concave up (== convex).

c. Prove: If $f(x) : [a, b] \to \Re$ is a differentiable function with the property that $f''(x) > 0$ for all $x \in [a, b]$, then show that $f(x)$ is concave up on $[a, b]$.

8.

a. Define what it means for a function $f(x); [a, b] \to \Re$ to be integrable. (i.e. Define the symbol $\int_a^b f(x)\ dx$.)

b. Using the DEFINITION of the integral, prove that $\int_0^2 x\ dx = 2$.

c. Give a careful statement of the Fundamental Theorem of Calculus.

d. Prove: The second part of the Fundamental Theorem of Calculus.