ADVANCED CALCULUS MAA4102

FINAL EXAM

FALL 2004

Name:

No calculators permitted during the exam.
Each problem is worth 25 points.
Explain all answers!

1.

a. Give a careful statement of what it means for a sequence to converge to a number L.

b. Using the DEFINITION of limit show that $lim_{n\to\infty}(4+\frac{7}{n^2})=4$.

c. Describe an algorithm used to approximate the cube root of 11. $(=11^{\frac{1}{3}})$

d. Compute the first three steps in the approximation of $11^{\frac{1}{3}}$.

2.

a. Give a careful statement of the least upper bound principle.

b. Prove that every bounded increasing sequence converges.

c. Prove the Bolzano-Weierstrass theorem. (i.e. Every bounded sequence has a convergent subsequence.)

3.

a. Give a careful definition of what it means for a sequence to be *Cauchy*.

b. Prove: If a sequence is Cauchy, then it is bounded.

c. Prove: If a sequence is Cauchy, then it converges.

4.

a. Give a careful statement of the Extremum Theorem.

b. Give a careful statement of Rolle's Theorem.

c. Prove Rolle's Theorem.

d. Show how Rolle's Theorem can be used to prove the Mean Value Theorem.

5.

a. Give a careful statement of the intermediate value theorem.

b. If $f(x) = x^5 + x + 1$, then show that $f(x)$ has at least one root in the interval $[-1, 1]$.

c. If $f(x) = x^5 + x + 1$, then show that $f(x)$ cannot have two distinct real roots.

6.

a. Give a careful statement of Taylor's theorem. (Be sure to include the error term.)

b. Give a careful statement of the Lagrange theorem for polynomial interpolation. (Be sure to include the error term.)

c. Given data $(0,2),(1,-3),(2,2),(3,-2),(4,2),(5,3)$ set up a matrix equation that will produce the 5^{th} degree polynomial $p_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ that will interpolate the data.

d. Given the function $f(x) = e^{\frac{1}{5}x}$ for $x \in [-2,2]$ and $tol = \frac{1}{10^7}$, find an integer n so that the Lagrange interpolation polynomial approximates $f(x)$ with error less than $\frac{1}{10^7}$ for all $x \in [-2,2]$.

7.

a. Prove: Limits are unique. (i.e. Prove: If $\lim_{\to\infty} x_n = L_1$ and $\lim_{\to\infty} x_n = L_2$, then $L_1 = L_2$.)

b. Give a careful statement of the n^{th} root test.

c. Prove the n^{th} root test.

8.

a. Prove: If $f(x)$ is differentiable, then $f(x)$ is continuous.

b. Define what it means for the integral to exist. (i.e. Define the symbol $\int_a^b f(x)\ dx$.)