L1 Precalc review

1. **Function:** rule which assigns to each element in set D exactly one element in set R.
   - domain = D (allowable inputs)
   - range = R (outputs)
   - graph: \( \{(x, y) : x \text{ is in } D \text{ and } y = f(x) \} \)
   - zero (root, solution): \( c \) is a zero of \( f(x) \) if \( f(c) = 0 \)
   - even/odd function even: \( f(-x) = f(x) \), odd: \( f(-x) = -f(x) \)
   - increasing/decreasing increasing: walk uphill from L to R, decreasing: walk downhill from L to R

2. **Transform** the graph of \( y = f(x) \)
   - \( f(x) \pm c \): shifts graph up/down \( c \) units
   - \( f(x + c) \): shifts graph left \( c \) units while \( f(x - c) \): shifts graph right \( c \) units
   - \( cf(x) \) is a vertical stretch if \( c > 1 \) and vertical shrink if \( 0 < c < 1 \)
   - \( f(cx) \): horizontal shrink if \( c > 1 \), horizontal stretch if \( 0 < c < 1 \)
   - \( -f(x) \): reflect across \( x \)-axis
   - \( f(-x) \): reflect across \( y \)-axis
Solving equations

Factor and solve \( x^{-\frac{3}{3}}(2 - x)^2 - 6x^{\frac{1}{3}}(2 - x) = 0 \)

Solve \( x = \sqrt{5 - x^2} - 1 \)

Solving inequalities:

\( a < b \) and \( c > 0 \) \( \rightarrow \) \( ac < bc \)

\( a < b \) and \( c < 0 \) \( \rightarrow \) \( ac > bc \)

Find the solution set: \( \frac{10-x}{x+2} \geq 2 \)
Absolute Value

Def. If $a$ is a real number $|a|$

so that $|a| = \begin{cases} 
    a \leq 0 \\
    a > 0
\end{cases}$

ex If $x \neq 0$, find an expression for $f(x) = \frac{x}{|x|}$

$\frac{x}{|x|} = \begin{cases} 
    \frac{x}{-x} & x < 0 \\
    \frac{x}{x} & x > 0
\end{cases}$

ex if $x \neq 1$ find an expression for

$g(x) = \frac{2(x-1)}{|x-1|} = \begin{cases} 
    x & x < 1 \\
    x & x > 1
\end{cases}$

Absolute Value Inequalities

Let $a > 0$:

$|x| < a$ if and only if

$|x| > a$ if and only if

ex. Solve and express your answer using intervals.

$3 - \left|\frac{1-3x}{2}\right| < -1$

Graph the solution set.
Elementary functions

1. **Polynomials** \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) where \( n \) is a nonnegative integer

   - linear functions \( f(x) = mx + b \)
      
      point-slope form: \( y - y_1 = m(x - x_1) \) uses point \((x_1, y_1)\) and slope \( m = \frac{y_2 - y_1}{x_2 - x_1} \)

      slope-intercept form: \( y = mx + b \), \( m = \text{slope}, (0, b) = \text{y-intercept} \)

   - quadratic functions \( f(x) = ax^2 + bx + c, a \neq 0 \)

      
      \( f(x) = a(x - h)^2 + k \) (complete the square to put in vertex form)

   - **power functions** \( f(x) = x^n, n \) a positive integer

2. **Rational functions** \( f(x) = \frac{P(x)}{Q(x)} \) where \( P \) and \( Q \) are polynomials

   - reciprocal function \( f(x) = \frac{1}{x} \)

3. **Root functions** \( f(x) = x^{\frac{1}{n}}, n \) is a positive integer

4. **Algebraic functions**: functions that can be constructed from polynomial functions using operations addition, subtraction, multiplication, division, and taking roots

5. **Transcendental functions**, which are not algebraic (trigonometric functions, inverse trig functions, exponential and logarithmic functions)
Domain

\[ f(x) = 2x + 1 \quad \quad g(x) = \frac{2x^2 - 3x - 2}{x - 2} \]

Are these equivalent functions?

Piecewise Defined Functions

ex. a) Sketch the graph:

\[ f(x) = \begin{cases} 
  x & x < 0 \\
  \frac{x}{|x|} & 0 < x < 2 \\
  3 - x & 0 < x < 2 \\
  2x^2 - 8x + 9 & x \geq 2 
\end{cases} \]

b) Find each interval on which \( f(x) \) is increasing, decreasing and constant.
Translations and Transformations

ex. \( y = 2 - \sqrt{x - 1} \)

Use the following order to graph a function involving more than one transformation:
1. Horizontal Translation
2. Stretching or shrinking
3. Reflecting
4. Vertical Translation (done last)
Function Composition

**Def.** \((f \circ g)(x) =\)

**Def.** \((g \circ f)(x) =\)

**ex.** If \(f(x) = \frac{1}{x + 2}\) and \(g(x) = \frac{4}{x - 1}\), find with domain: 

\((f \circ g)(x)\)

**NOTE:** The domain of \(F(x) = (f \circ g)(x)\) is the intersection of the domain of inner function \(g\) and the resulting function \(F\).
Trigonometry

1. Unit conversion: degrees $\leftrightarrow$ radians

2. Trigonometric functions

\[
\begin{align*}
\sin \theta &= \\
\cos \theta &= \\
\tan \theta &= \\
\csc \theta &= \\
\sec \theta &= \\
\cot \theta &= 
\end{align*}
\]

3. Two basic triangles

4. Unit circle ($r = 1$, so $\sin \theta = y$ and $\cos \theta = x$)

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>$\pi/6$</th>
<th>$\pi/4$</th>
<th>$\pi/3$</th>
<th>$\pi/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5. Graphs

\[ y = \sin x \]

\[ y = \cos x \]

\[ y = \tan x \]

\[ y = \cot x \]

\[ y = \sec x \]

\[ y = \csc x \]

**NOTE:** \[ \leq \sin x \leq 1 \], \[ \leq \cos x \leq 1 \]
Identities

1) $\sin^2 \theta + \cos^2 \theta = 1$

2) $\tan^2 \theta + 1 = \sec^2 \theta$

3) $1 + \cot^2 \theta = \csc^2 \theta$

4) $\sin(-\theta) = -\sin \theta$

5) $\cos(-\theta) = \cos \theta$

Add/subtract formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Double Angle formulas

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

Half-Angle formulas

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

ex. Solve for $\theta$ in $[0, 2\pi)$ if $\sqrt{3} \sin 2\theta + 2 \sin^2 \theta = 0$. 

Inverse functions.

1. One-to-one functions

**Def.** A function $f$ is called a **one-to-one function** if for any $x_1$ and $x_2$ in the domain:

   if $x_1 \neq x_2$ then

   • Horizontal Line Test

2. Inverse functions

   $f^{-1}(y) = x$ if and only if

   • If $(x, y)$ is a point on the graph of $f(x)$, then

   Therefore, the graph of $f^{-1}$ is the graph of $f$ reflected through the line

   • domain and range of $f^{-1}$

   • inverse relationships

   $f^{-1}(f(x)) = x$ for every $x$ in $A$

   $f(f^{-1}(x)) = x$ for every $x$ in $B$

**ex.** Find the inverse of $f(x) = \sqrt{x + 2}$. Check domain and range.
Inverse Trigonometric Functions

- $y = \sin^{-1} x$ if and only if

- $y = \cos^{-1} x$ if and only if

- $y = \tan^{-1} x$ if and only if

There are similar definitions for the inverse of the other trigonometric functions.

**NOTE: Inverse Properties**

1. $\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$
   
   $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

2. $\cos(\cos^{-1} x) = x$ for $-1 \leq x \leq 1$
   
   $\cos^{-1}(\cos x) = x$ for $0 \leq x \leq \pi$

3. $\tan(\tan^{-1} x) = x$ for all $x$
   
   $\tan^{-1}(\tan x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$
ex. Find the following if possible:

1) $\sin^{-1}\left(\frac{-1}{2}\right)$
2) $\cos^{-1}\left(\frac{-1}{2}\right)$
3) $\cos^{-1}(2)$

4) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$
5) $\tan^{-1}\left(\tan\frac{7\pi}{5}\right)$

ex. Use a triangle to find the exact value: $\sin(\tan^{-1}(-2))$

ex. Use a triangle to simplify the expression: $\cos(2\tan^{-1}x)$
Exponential Functions

**Def.** An exponential function with base $b$ is a function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

- **Laws of Exponents ($b \neq 0$)**
  1.) $b^0 =$
  2.) $b^{-x} =$
  3.) $b^{1/n} =$ \hspace{2cm} $n$ a positive integer
  4.) $b^x \cdot b^y =$
  5.) $\frac{b^x}{b^y} =$
  6.) $(b^x)^y =$
  7.) $(ab)^x =$

- If $b > 0$ and $b \neq 1$, then $y = b^x$ is a one-to-one increasing or decreasing function.

\[
f(x) = b^x \quad \quad f^{-1}(x) = \log_b(x)
\]

1. domain:

2. range:

3. intercept:

4. asymptote:

5. increasing if
decreasing if
Logarithmic functions

The inverse of $y = b^x$ is the logarithmic function with base $b$, written

$y = \log_b(x)$ if and only if

- Laws of Logarithms ($x > 0$ and $y > 0$)
  1.) $\log_b(1) =$
  2.) $\log_b(b) =$
  3.) $\log_b(xy) =$
  4.) $\log_b \left( \frac{x}{y} \right) =$
  5.) $\log_b(x^n) =$

- Inverse Properties
  1.) $\log_b(b^x) =$
  2.) $b^{\log_b x} =$

- Change of base formula
  For any $b > 0$ and $b \neq 1$, $\log_b(x) = \frac{\log_a x}{\log_a b}$

Properties of graphs of $f(x) = b^x$ and $f^{-1}(x) = \log_b(x)$:

$\begin{array}{c}
\text{Properties of graphs of } f(x) = b^x \text{ and } f^{-1}(x) = \log_b(x): \\
\text{For any } b > 0 \text{ and } b \neq 1, \log_b(x) = \frac{\log_a x}{\log_a b}
\end{array}$

There is a base that occurs frequently in applications and is especially useful in calculus. It is the irrational number $e \approx 2.7182818284590452353602874713526625...$
**Def.** Common logarithm: $\log x =$

**Def.** Natural logarithm: $\ln x =$

**ex.** Evaluate:

1) $e^{-3 \ln 2} \\
2) \log \left( \frac{1}{1000^2} \right)$

**ex.** Solve for $x$:

$$2 \ln(x) - \ln(3 - x) = \ln \left( \frac{1}{2} \right) + \ln(8)$$

**ex.** Find the inverse of $f(x) = \log(x + 2) - 1$. 