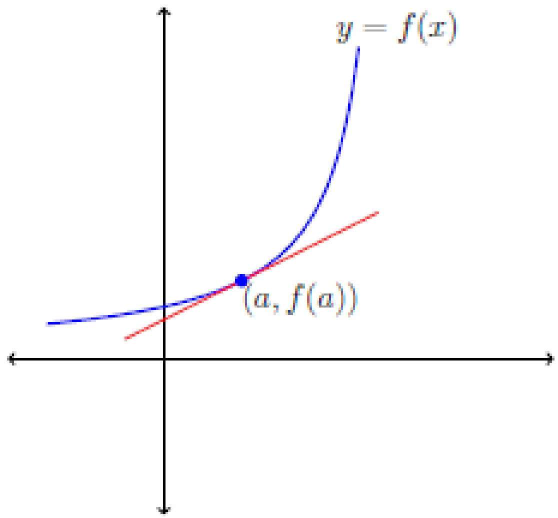
L10 Linear Approximations

We have seen that a curve $y = f(x)$ closely follows its tangent line at a point $(a, f(a))$. That is, we can approximate a function value $f(x)$ for x near a by the corresponding function value on the tangent line.

Find the equation of the tangent line to $y = f(x)$ at $(a, f(a))$:



The linearization of f at a is the linear function

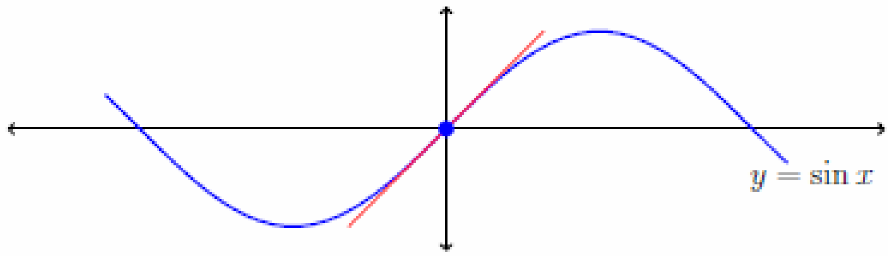
$$L(x) =$$

For x-values near a, we note that

This is the **Tangent line** or **Linear approximation** of f at a.

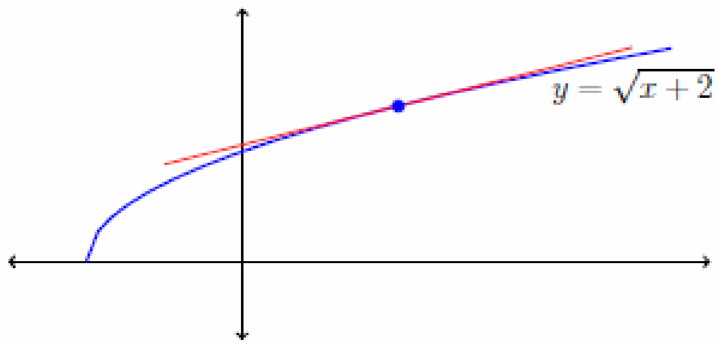
ex. Find the linearization of $f(x) = \sin x$ at $a = 0$.

Use it to approximate $\sin \frac{\pi}{6}$ and $\sin \frac{\pi}{10}$.



NOTE: $\sin \frac{\pi}{6} = 0.5$ and $\sin \frac{\pi}{10} = 0.3090...$

ex. Find the linearization of $f(x) = \sqrt{x+2}$ at $a = 2$, and use it to approximate $\sqrt{3.96}$ and $\sqrt{4.04}$.



NOTE: $\sqrt{3.96} = 1.989975...$ and $\sqrt{4.04} = 2.009975...$

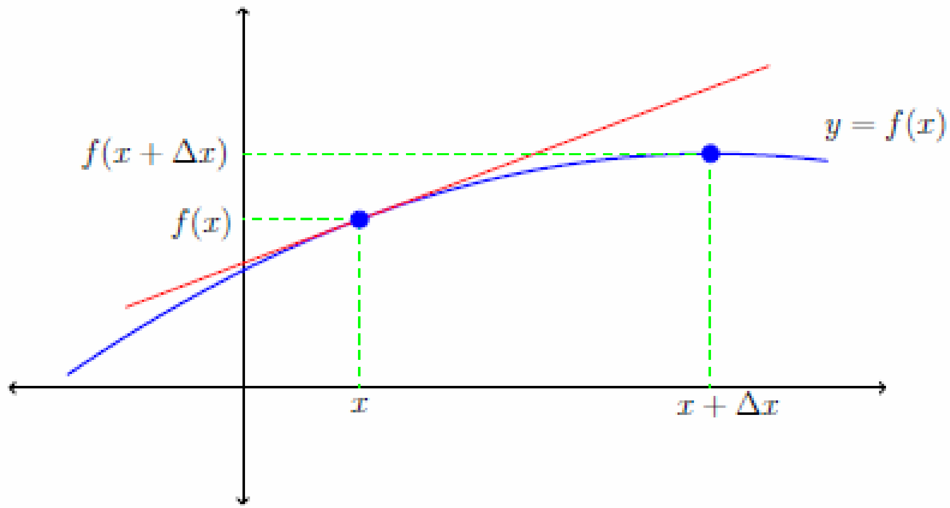
Differentials

<u>Def.</u> Let $y = f(x)$ be a differentiable function.
The **differential** dx is an independent variable (it can be any real number). The **differential** dy or $d(f(x))$ is a dependent variable defined by

<u>ex.</u> 1) Find dy if $y = f(x) = \dfrac{x}{3 - x^2}$.

2) Evaluate dy when $x = 2$ and $dx = -0.02$.

Geometrically



$f(x + \Delta x) =$

$f(x + \Delta x) \approx$

ex. Compare Δy and dy if $y = f(x) = \dfrac{x}{3 - x^2}$ and x changes from 2 to 1.98.

NOTE: Linearization and the differential

We have seen that for an x-value near a number a,

$f(x) \approx L(x) = f(a) + f'(a)(x - a)$.

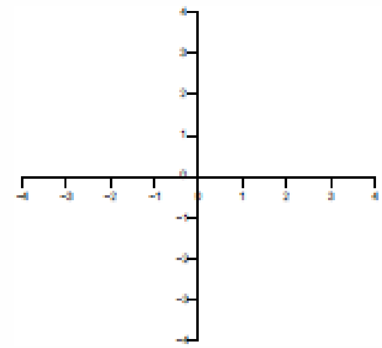
<u>ex.</u> Use differentials to approximate $\sqrt[3]{7.8}$.

Actual value: $\sqrt[3]{7.8} = 1.983129248...$

Extreme Values

<u>Def.</u> A function f has an **absolute maximum** at $x = c$ if

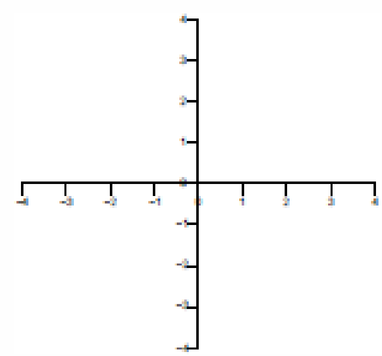
$f(x) = 2 - |x|$



$f(c)$ is the   of f on D.

<u>Def.</u> f has an **absolute minimum** at $x = c$ if

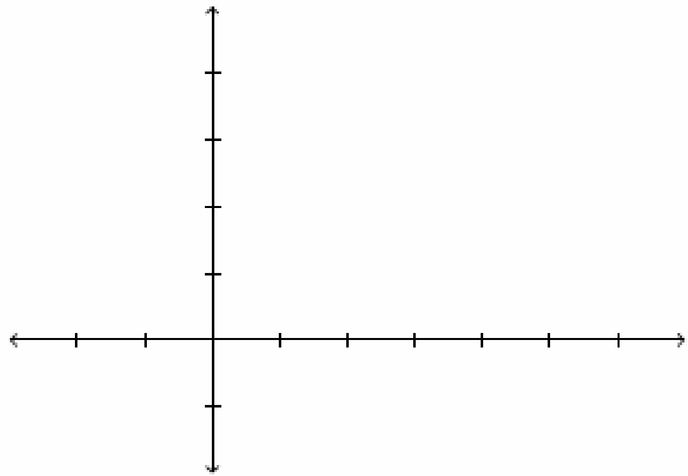
$f(x) = x^2 - 1$



$f(c)$ is the   of f on D.

Together they are called

Let $f(x) = \begin{cases} x^2 & -2 \le x \le 1 \\ \sqrt{x-1} & x > 1 \end{cases}$.



Find the absolute maximum and minimum values of $f(x)$ on each of the following intervals:

$[-2, 1]$

$[2, 5]$

$(-2, 1)$

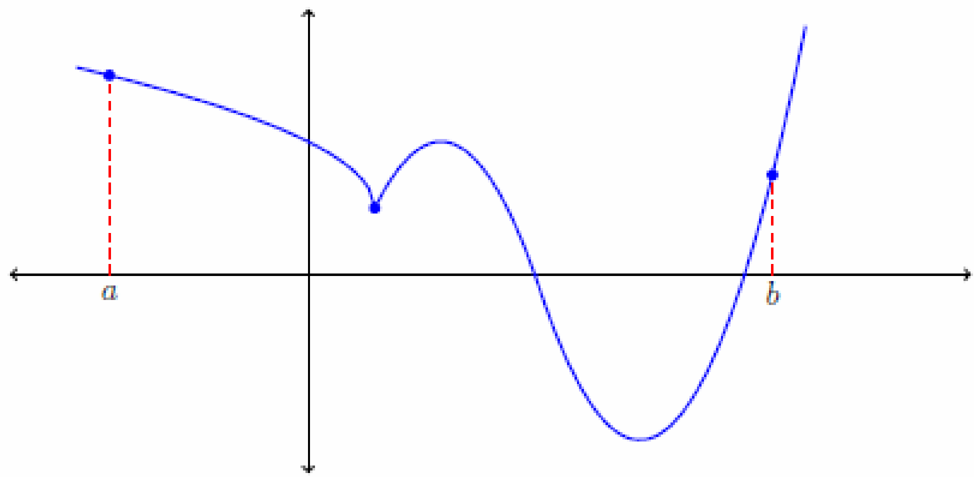
$(1, \infty)$

Are there conditions for which absolute extrema must exist?

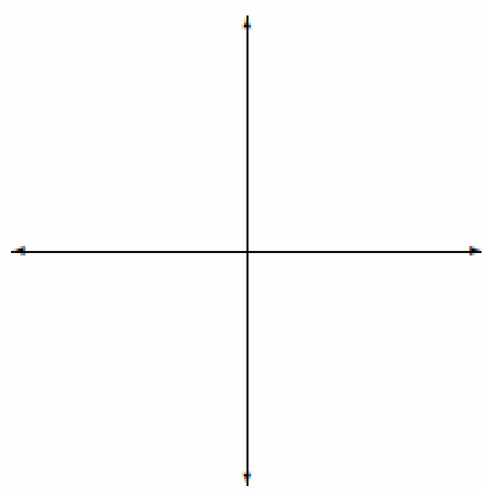
Extreme Value Theorem:

If f is continuous on a closed interval $[a, b]$, then f has

at some numbers c and d in $[a, b]$.



ex. Find the absolute extrema of $f(x) = \dfrac{1}{x}$ on $[-1, 1]$.

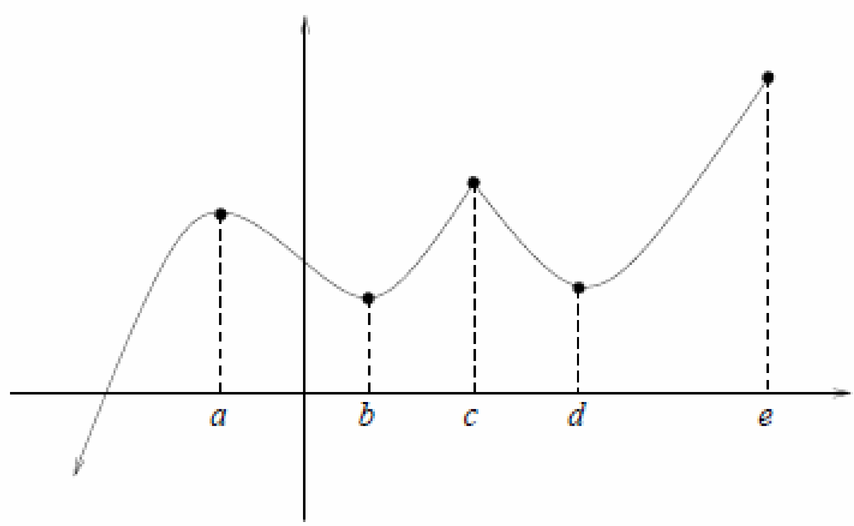


How can we find those extreme values?

Def. A function f has a local (relative) maximum at c if there is an open interval I containing c such that for all x in I

Def. A function f has a local (relative) minimum at c if there is an open interval I containing c such that for all x in I

ex. Find all local and absolute extrema of the function $f(x)$ sketched below.



Where might a local extreme value occur?

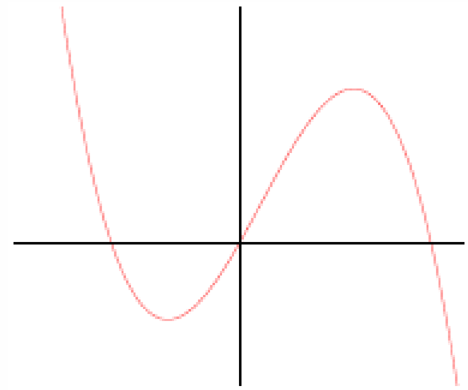
Def. A number c in the domain of a function f is called a **critical number** if

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ does not exist.}$$

ex. Find all critical numbers of $f(x) = \dfrac{x^{2/3}}{x+1}$.

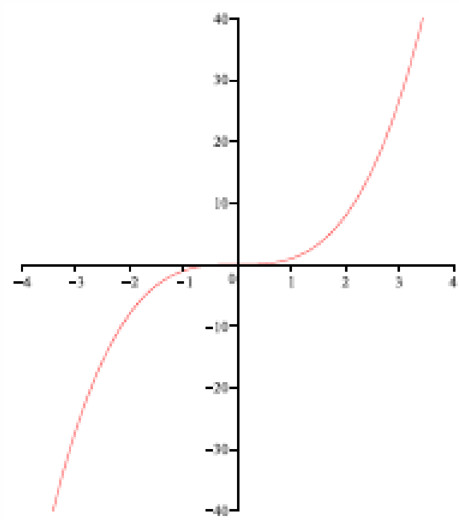
Note that $f'(x) = \dfrac{2-x}{3x^{1/3}(x+1)^2}$.

Fermat's Theorem:
If f has a local extremum at $x = c$, and if $f'(c)$ exists, then

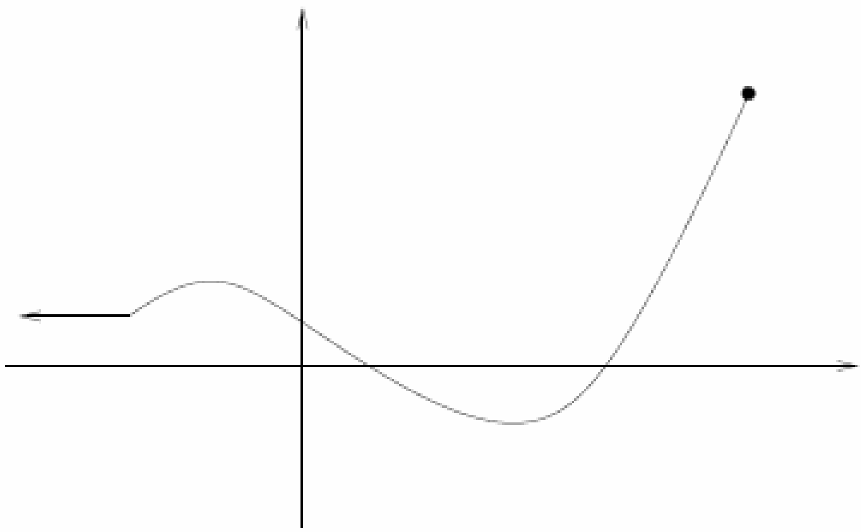


NOTE: If f has a local extreme value at c, then c is a

<u>**ex.**</u> Find the critical numbers and local extrema of $f(x) = x^3$.



We can now see that the absolute extrema of a function $f(x)$ must occur at one of two places:



1) 2)

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1.

2.

3.

ex. Find the maximum and minimum values of $f(x) = 2x^{5/3} - 5x^{2/3}$ on $[0, 8]$.

ex. Find the absolute maximum and minimum values of

$f(x) = \dfrac{\ln x}{x}$ on the intervals

1) $[1, e^2]$

2) $\left[\dfrac{1}{e}, 1\right]$