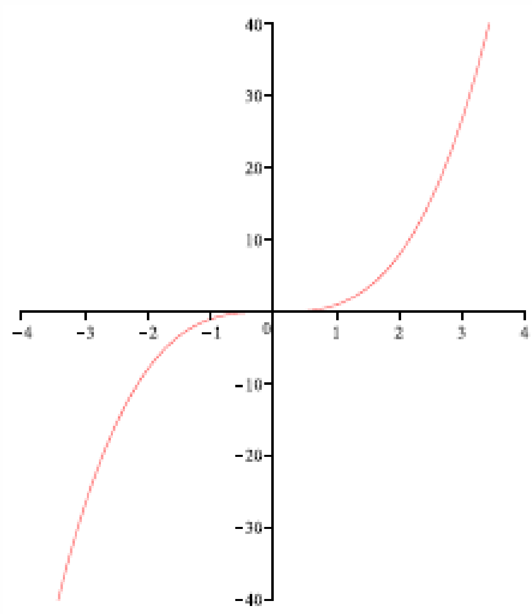
L12 Second derivative test and curve sketching

Concavity



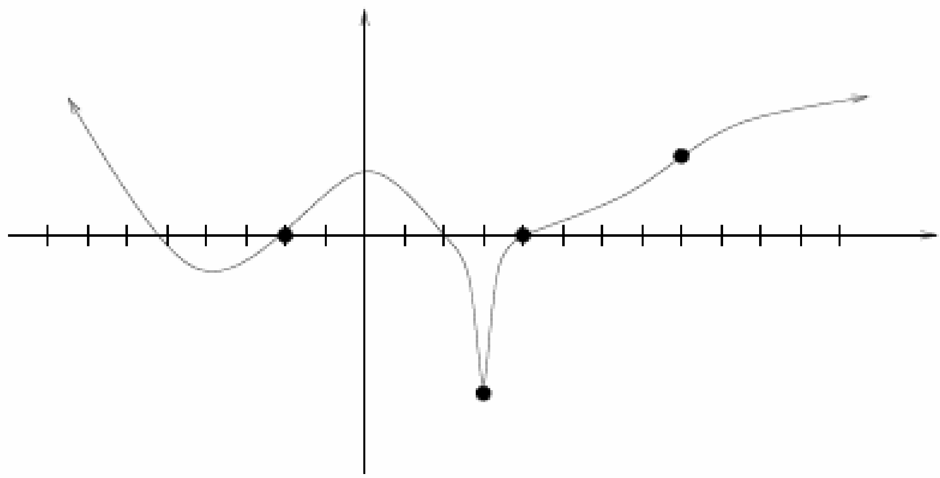
$$f(x) = x^3$$

<u>**Def.**</u> If $f(x)$ is differentiable on (a, b), then

1. f is **concave up** on (a, b) if

2. f is **concave down** on (a, b) if

Consider the graph of $f(x)$ sketched below:



Where is the function $f(x)$ concave up and down?

Test for Concavity

Assume that $f''(x)$ exists on (a, b).

1. If $f''(x) > 0$ for all x on (a, b), then f

2. If $f''(x) < 0$ for all x on (a, b), then f

Def. A point $P = (c, f(c))$ called an **inflection point** if

$$f(x) \text{ is continuous} \quad \text{and} \quad f''(x) \text{ changes sign}$$

at $x = c$.

ex. Find each interval on which the graph of $f(x) = \dfrac{x}{x^2+1}$ is concave up and down. Find each inflection point of $f(x)$.

Note that $f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$.

Derivatives and the Shape of a Graph

	$f' > 0$	$f' < 0$
$f'' > 0$		
$f'' < 0$		

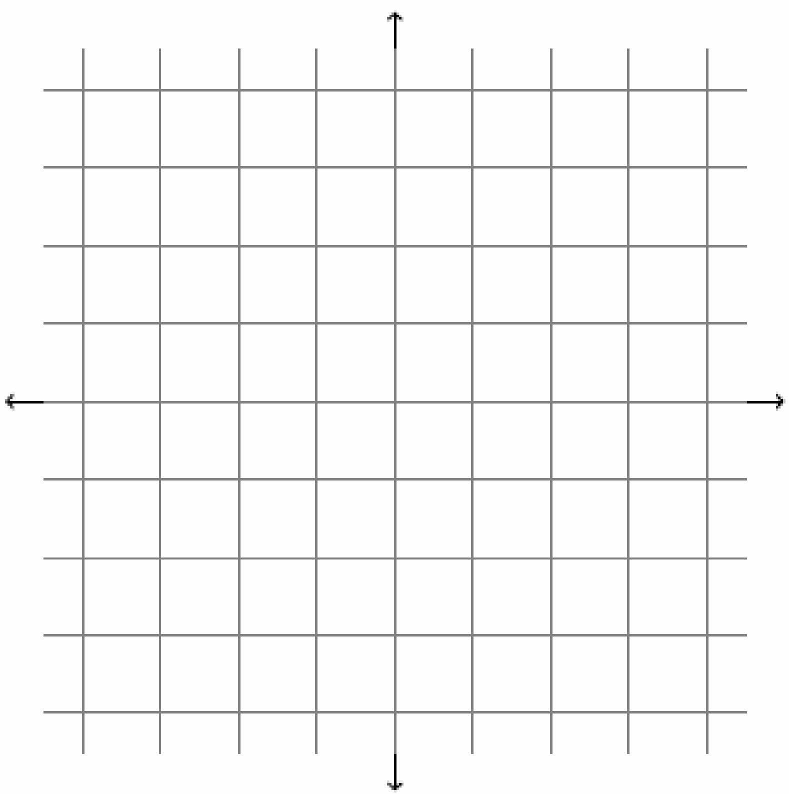
ex. Find all relative extrema and inflection points of the graph of $f(x) = 2x^{5/3} - 5x^{2/3}$.

1. $f(x)$:

2. $f'(x) = \dfrac{10}{3}x^{2/3} - \dfrac{10}{3}x^{-1/3}$

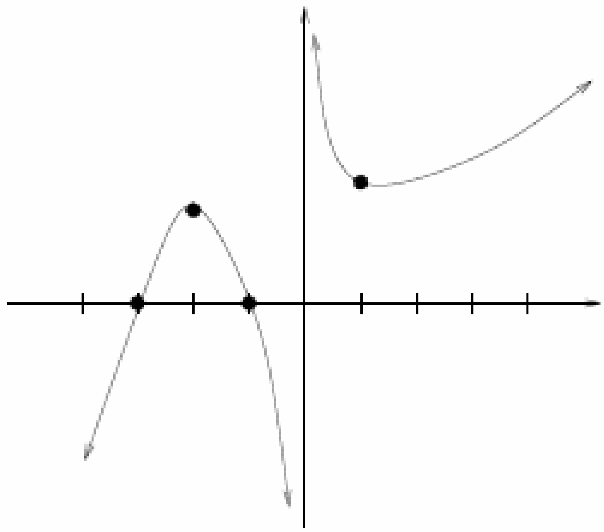
3. $f''(x) = \dfrac{20x + 10}{9x^{\frac{4}{3}}}$.

Sketch the graph of $f(x) = 2x^{5/3} - 5x^{2/3}$.



$$f\left(-\frac{1}{2}\right) \approx -3.78$$

ex. Suppose that f is continuous on $(-\infty, \infty)$. Given the graph of $f'(x)$, find the following:



1. Intervals on which f is increasing/decreasing

2. Local extrema

3. Intervals on which f is concave up/down

4. Points of inflection

Second Derivative Test

Suppose that $f''(x)$ is continuous near c.

1. If $f'(c) = 0$ and $f''(c) < 0$, then

2. If $f'(c) = 0$ and $f''(c) > 0$, then

<u>**ex.**</u> Use the Second Derivative Test if possible to find the local (relative) extrema of $f(x) = 3x^5 - 5x^3$.

To Sketch the Graph of $y = f(x)$

1. Use $f(x)$ for finding:
   - Domain
   - Intercepts
   - Symmetry
   - Asymptotes
2. Use $f'(x)$ for finding:
   - Critical Numbers
     (horizontal tangents, vertical tangents/cusps)
   - Increasing/Decreasing Intervals
   - Local Extrema
3. Use $f''(x)$ for finding:
   - Concavity
   - Points of Inflection

<u>ex.</u> Sketch the graph of $f(x) = x^{\frac{1}{3}}(x-4)$.

1.

2. $f'(x) = \dfrac{4x-4}{3x^{\frac{2}{3}}}$

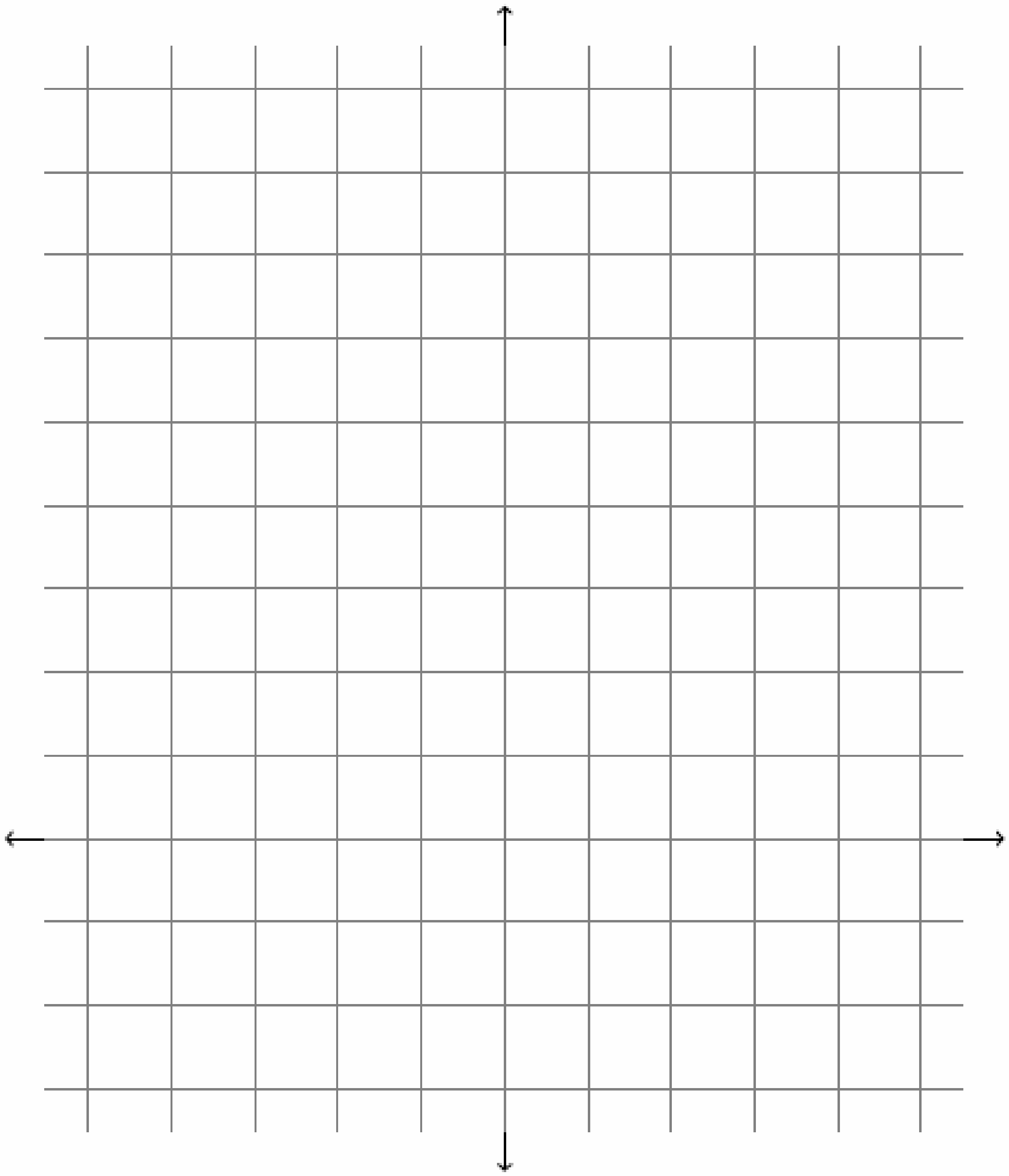
f'

3. $f''(x) = \dfrac{4x+8}{9x^{\frac{5}{3}}}$

f''

4. Shape of graph:

f



$$f(-2) \approx 7.6$$

ex. Sketch the graph of $f(x) = \dfrac{x}{(x-2)^2}$.

1.

2. $f'(x) = \dfrac{-(x+2)}{(x-2)^3}$

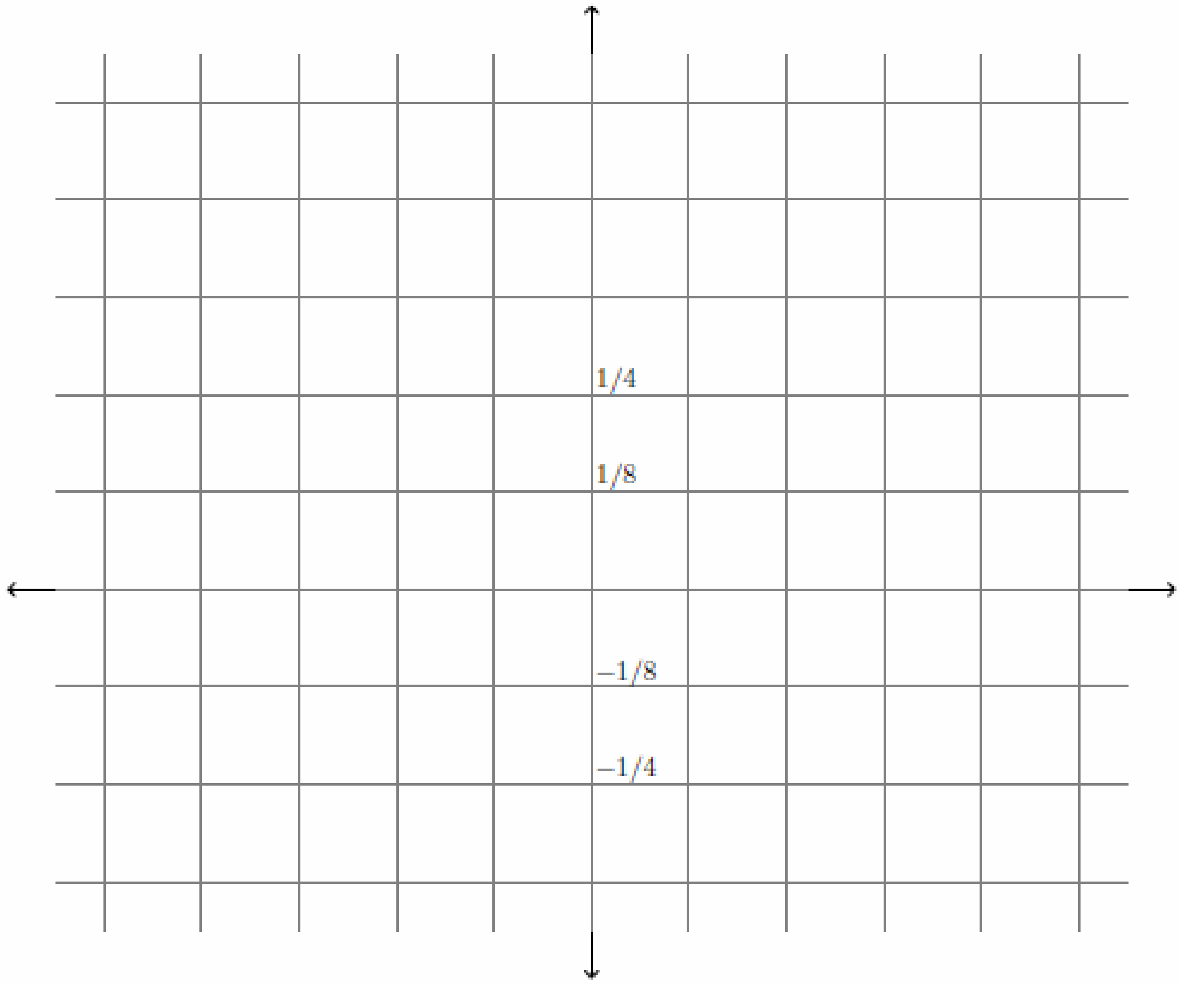
f' $\longleftrightarrow$

3. $f''(x) = \dfrac{2(x+4)}{(x-2)^4}$

f'' $\longleftrightarrow$

4. Shape of graph:

f $\longleftrightarrow$



$$f(-4) = -\frac{1}{9} \text{ and } f(-2) = -\frac{1}{8}$$

ex. Sketch the graph of $f(x) = \dfrac{\ln x}{x}$.

1.

2. $f'(x) = \dfrac{1 - \ln x}{x^2}$

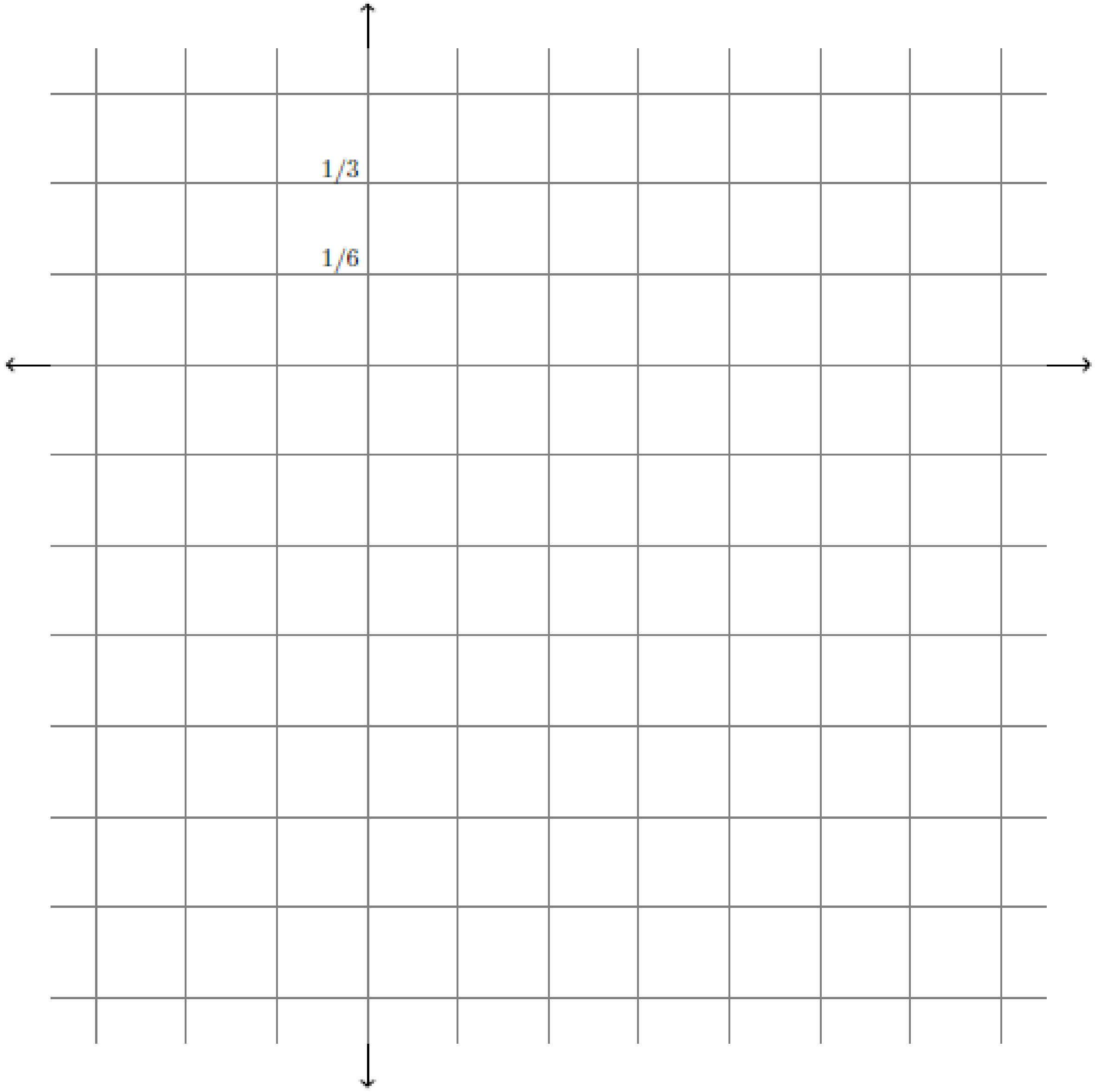
f'

3. $f''(x) = \dfrac{2 \ln x - 3}{x^3}$

f''

4. Shape of graph:

f



$$f(e) \approx 0.37$$

$$e^{3/2} \approx 4.5 \text{ and } f\left(e^{3/2}\right) \approx 0.33$$

ex. Sketch the graph of $f(x) = e^{-\frac{1}{x}}$.

1.

2. $f'(x) = \dfrac{e^{-\frac{1}{x}}}{x^2}$

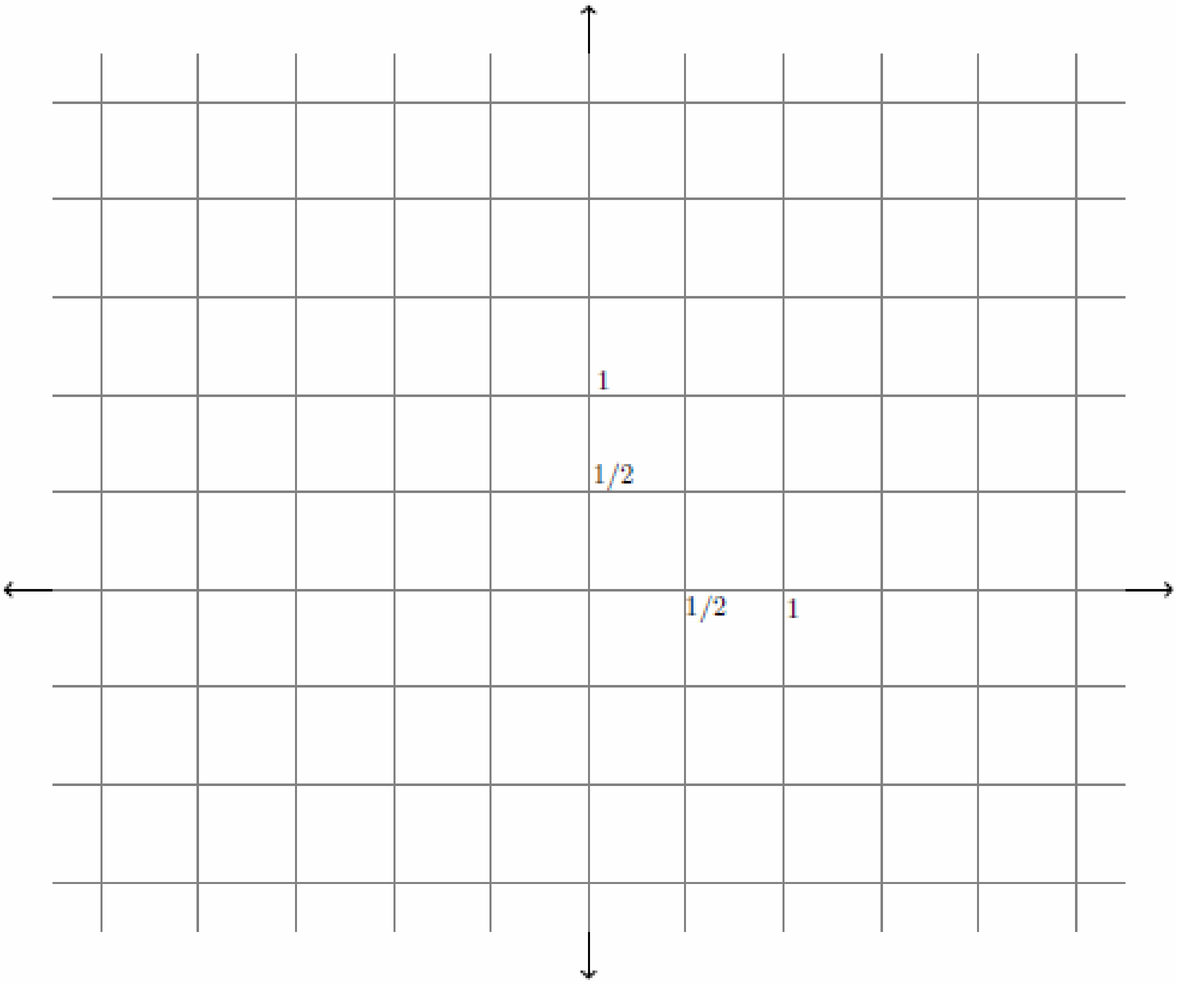
f'

3. $f''(x) = \dfrac{e^{-\frac{1}{x}}(1-2x)}{x^4}$

f''

4. Shape of graph:

f



$$f\left(\frac{1}{2}\right) \approx 0.14$$