L14 L'Hopital's Rule and Antiderivatives

Consider the following limits:

ex. $\lim_{x \to 0} \frac{e^x - 1}{x}$

ex. $\lim_{x \to \infty} \frac{\ln(x)}{x+1}$

These limits are **Indeterminate Forms** $\frac{0}{0}$ and $\frac{\infty}{\infty}$. We have seen how to evaluate some limits of this form, but the process can be complicated.

ex. $\lim_{x \to 1} \frac{\sqrt{x^2+3} - 2}{x - 1}$

$\lim_{x \to 0} \frac{\sin x}{x}$

We introduce a new method that applies to many limits of this form:

L'Hôpital's Rule

Suppose that f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty.$$

Then

if it exists or is $\pm\infty$.

Note the following:

1. L'Hôpital's rule applies to one-sided limits and limits at infinity.

2. L'Hôpital's rule is the limit of quotient of derivatives, <u>not</u> the limit of derivative of the quotient.

ex. Find the following limits:

1. $\lim_{x \to 0} \dfrac{a^x - 1}{x}$

2. $\lim_{x \to 0} \dfrac{\sin x}{x}$

3. $\lim_{x \to 0^-} \dfrac{\tan x}{x^2}$

4. $\lim_{x \to 0} \dfrac{e^{3x} - 3x - 1}{x^2}$

ex. Find the following limits:

1. $\displaystyle \lim_{x\to+\infty} \frac{\ln(x^2+1)}{\ln x}$

2. $\displaystyle \lim_{x\to 0^+} \frac{\ln x}{\csc x}$

Caution: You must have an indeterminate form to use L'Hôpital's Rule!

ex. Find the limit: $\lim_{x \to \frac{\pi}{2}} \frac{\cos x + x}{\sin x}$

ex. Evaluate: $\lim_{x \to 0^+} \frac{1 + e^x}{\ln x}$

Indeterminate Products $(0 \cdot \infty)$

ex. Evaluate:

$$\lim_{x \to 0} \left(\frac{1}{x} \cdot x \right) =$$

$$\lim_{x \to 0} \left(\frac{1}{x} \cdot x^2 \right) =$$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm\infty$, how to find $\lim_{x \to a} f(x)\, g(x)$?

Rewrite as

ex. Find $\lim_{x \to 0^+} x \ln(x)$

ex. Find $\lim_{x \to \frac{\pi}{4}} (1 - \tan x) \sec 2x$.

Indeterminate Differences $(\infty - \infty)$

ex. Consider the following:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x}\right) = \qquad\qquad \lim_{x\to 0}\left(\frac{2}{x}-\frac{1}{x}\right) =$$

ex. $\displaystyle \lim_{x\to 0^+}\left(\frac{1}{x}-\csc x\right)$

Indeterminate Powers (0^0 , ∞^0 , or 1^∞)

To evaluate, rewrite $\lim\limits_{x \to a}[f(x)]^{g(x)}$ as

$$\lim_{x \to a} e^{\ln([f(x)]^{g(x)})} = e^{\left(\lim\limits_{x \to a} g(x) \ln f(x)\right)}.$$

That limit has the form $0 \cdot \infty$.

<u>**ex.**</u> Evaluate: $\lim\limits_{x \to 0}(1+x)^{\frac{1}{x}}$

ex. Find $\lim_{x \to 0^+} (1 + \sin 2x)^{\cot x}$.

Antiderivatives (Section 4.9)

ex. Suppose that the slope of the tangent line of a function $f(x)$ at any x-value is given by $2x - 5$. Can we find $f(x)$?

Def. A function F is called an of f on an interval I if for all x in I.

ex. Find an antiderivative of the function $f(x) = x^4 + x$.

How many antiderivatives can there be?

Recall the theorem (page L24 - 8) which states that if two functions have the same derivative on an interval, they can only differ by a constant.

Theorem: If F is an antiderivative of f on I, then

is the most general antiderivative of f on I, where C represents any constant.

ex. Find the most general antiderivative of the following:

1) $f(x) = \sec x \tan x$

2) $f(x) = e^{5x}$

NOTE: If $f(x) = x^n$ $(n \neq -1)$, then $F(x) =$

If $n \geq 0$, then x

If $n < 0$, then x

ex. Find $F(x)$ if $f(x) = x^{-3}$.

ex. If $f(x) = \dfrac{1}{x}$, find $F(x)$.

Antidifferentiation Formulas

Let F and G be antiderivatives of functions f and g.

Function	General antiderivative
$cf(x)$	
$f(x) \pm g(x)$	
$x^n \ (n \neq -1)$	
$\frac{1}{x}$	
e^x	
$\sin x$	
$\cos x$	
$\sec^2 x$	
$\sec x \tan x$	
$\csc^2(x)$	
$\csc x \cot x$	
$\frac{1}{\sqrt{1-x^2}}$	
$\frac{1}{1+x^2}$	

Notation: $\int f(x)\,dx = F(x) + C$ means that

We say that $F(x) + C$ is the general antiderivative or **indefinite integral** of $f(x)$.

ex. Find all functions $f(x)$ such that

$$f'(x) = 6x^3 - 3\sqrt{x} - \frac{1}{\sqrt{1-x^2}}.$$

ex. Find the general antiderivative of each of the following:

1) $f(x) = \tan^2 x + 1$

2) $f(x) = \dfrac{x^2+3}{x^2+1}$

__ex.__ Find all functions $g(x)$ such that

$$g'(x) = \frac{1}{x} + \frac{3}{x^2} - 6.$$

What can you say about the graphs of those functions?

Particular Solutions

__ex.__ Find $f(x)$ if $f'(x) = \sin x + 2$ and $f(\pi) = -1$.