Consider the following limits:

\[
\text{ex. } \lim_{x \to 0} \frac{e^x - 1}{x}
\]

\[
\text{ex. } \lim_{x \to \infty} \frac{\ln(x)}{x + 1}
\]

These limits are **Indeterminate Forms** \(\frac{0}{0}\) and \(\frac{\infty}{\infty}\). We have seen how to evaluate some limits of this form, but the process can be complicated.

\[
\text{ex. } \lim_{x \to 1} \frac{\sqrt{x^2 + 3} - 2}{x - 1}
\]

\[
\lim_{x \to 0} \frac{\sin x}{x}
\]
We introduce a new method that applies to many limits of this form:

**L’Hôpital’s Rule**

Suppose that $f$ and $g$ are differentiable and $g'(x) \neq 0$ near $a$ (except possibly at $a$). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty.$$

Then

if it exists or is $\pm \infty$.

Note the following:

1. L’Hôpital’s rule applies to one-sided limits and limits at infinity.

2. L’Hôpital’s rule is the limit of quotient of derivatives, **not** the limit of derivative of the quotient.
ex. Find the following limits:

1. \( \lim_{x \to 0} \frac{a^x - 1}{x} \)

2. \( \lim_{x \to 0} \frac{\sin x}{x} \)

3. \( \lim_{x \to 0^-} \frac{\tan x}{x^2} \)

4. \( \lim_{x \to 0} \frac{e^{3x} - 3x - 1}{x^2} \)
ex. Find the following limits:

1. \[ \lim_{x \to +\infty} \frac{\ln(x^2 + 1)}{\ln x} \]

2. \[ \lim_{x \to 0^+} \frac{\ln x}{\csc x} \]
Caution: You must have an indeterminate form to use L’Hôpital’s Rule!

**ex.** Find the limit: \( \lim_{x \to \frac{\pi}{2}} \frac{\cos x + x}{\sin x} \)

**ex.** Evaluate: \( \lim_{x \to 0^+} \frac{1 + e^x}{\ln x} \)
Indeterminate Products \((0 \cdot \infty)\)

**ex.** Evaluate:

\[
\lim_{x \to 0} \left( \frac{1}{x} \cdot x \right) = \quad \lim_{x \to 0} \left( \frac{1}{x} \cdot x^2 \right) =
\]

If \(\lim_{x \to a} f(x) = 0\) and \(\lim_{x \to a} g(x) = \pm \infty\), how to find \(\lim_{x \to a} f(x) g(x)\)?

Rewrite as

**ex.** Find \(\lim_{x \to 0^+} x \ln(x)\)
ex. Find \( \lim_{x \to \frac{\pi}{4}} (1 - \tan x) \sec 2x \).
Indeterminate Differences \((\infty - \infty)\)

**ex.** Consider the following:

\[
\lim_{{x \to 0}} \left( \frac{1}{x} - \frac{1}{x} \right) = \quad \lim_{{x \to 0}} \left( \frac{2}{x} - \frac{1}{x} \right) =
\]

**ex.** \[
\lim_{{x \to 0^+}} \left( \frac{1}{x} - \csc x \right)
\]
Indeterminate Powers \((0^0, \, \infty^0, \, \text{or} \, 1^\infty)\)

To evaluate, rewrite \(\lim_{x \to a} [f(x)]^{g(x)}\) as

\[
\lim_{x \to a} e^{\ln([f(x)]^{g(x)})} = e^{\left(\lim_{x \to a} g(x) \ln f(x)\right)}.
\]

That limit has the form \(0 \cdot \infty\).

\textbf{ex.} Evaluate: \(\lim_{x \to 0} (1 + x)^{\frac{1}{x}}\)
ex. Find \( \lim_{x \to 0^+} (1 + \sin 2x)^{\cot x} \).
Antiderivatives (Section 4.9)

**ex.** Suppose that the slope of the tangent line of a function $f(x)$ at any $x$-value is given by $2x - 5$. Can we find $f(x)$?

**Def.** A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

**ex.** Find an antiderivative of the function $f(x) = x^4 + x$.

How many antiderivatives can there be?

Recall the theorem (page L24 - 8) which states that if two functions have the same derivative on an interval, they can only differ by a constant.

**Theorem:** If $F$ is an antiderivative of $f$ on $I$, then

is the most general antiderivative of $f$ on $I$, where $C$ represents any constant.
ex. Find the most general antiderivative of the following:

1) \( f(x) = \sec x \tan x \)

2) \( f(x) = e^{5x} \)

**NOTE:** If \( f(x) = x^n \quad (n \neq -1) \), then \( F(x) = \)

- If \( n \geq 0 \), then \( x \)
- If \( n < 0 \), then \( x \)

ex. Find \( F(x) \) if \( f(x) = x^{-3} \).

ex. If \( f(x) = \frac{1}{x} \), find \( F(x) \).
**Antidifferentiation Formulas**

Let $F$ and $G$ be antiderivatives of functions $f$ and $g$.

<table>
<thead>
<tr>
<th>Function</th>
<th>General antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$cf(x)$</td>
<td></td>
</tr>
<tr>
<td>$f(x) \pm g(x)$</td>
<td></td>
</tr>
<tr>
<td>$x^n \ (n \neq -1)$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td></td>
</tr>
<tr>
<td>$e^x$</td>
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<tr>
<td>$\sin x$</td>
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<tr>
<td>$\cos x$</td>
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</tr>
<tr>
<td>$\sec^2 x$</td>
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<tr>
<td>$\sec x \tan x$</td>
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</tr>
<tr>
<td>$\csc^2(x)$</td>
<td></td>
</tr>
<tr>
<td>$\csc x \cot x$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1 - x^2}}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{1 + x^2}$</td>
<td></td>
</tr>
</tbody>
</table>
Notation: \( \int f(x) \, dx = F(x) + C \) means that

We say that \( F(x) + C \) is the general antiderivative or indefinite integral of \( f(x) \).

**ex.** Find all functions \( f(x) \) such that

\[
f'(x) = 6x^3 - 3\sqrt{x} - \frac{1}{\sqrt{1 - x^2}}.
\]

**ex.** Find the general antiderivative of each of the following:

1) \( f(x) = \tan^2 x + 1 \)

2) \( f(x) = \frac{x^2 + 3}{x^2 + 1} \)
ex. Find all functions $g(x)$ such that

$$g'(x) = \frac{1}{x} + \frac{3}{x^2} - 6.$$ 

What can you say about the graphs of those functions?

**Particular Solutions**

ex. Find $f(x)$ if $f'(x) = \sin x + 2$ and $f(\pi) = -1$. 