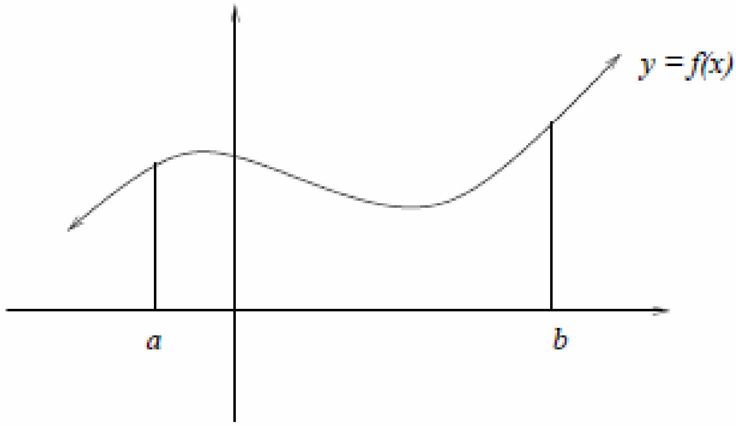
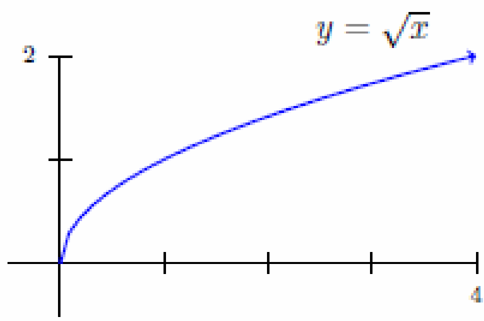
L15 Areas



How to find the area of the region that lies under the curve $y = f(x)$ from a to b?

<u>**ex.**</u> Let $f(x) = \sqrt{x}$ and consider the area beneath the graph of the function on $[0, 4]$.

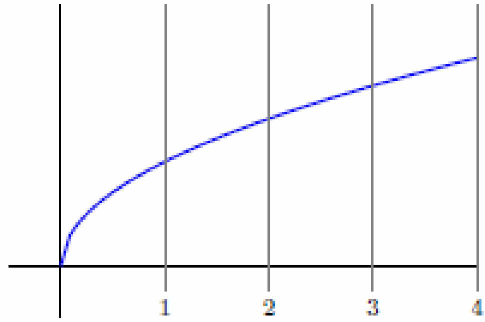


Let R_n be the sum of the areas of n rectangles with

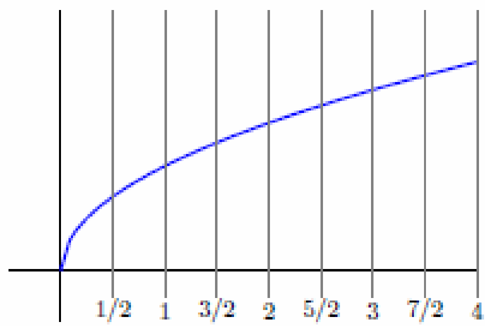
equal width

and height

1) Find R_4 ($n = 4$):



2) Find R_8 ($n = 8$):



For any n,

$R_n =$

As $n \to \infty$, what happens to our approximation?

We define the area as $\lim_{n\to\infty} R_n$

$$= \lim_{n\to\infty} \frac{4}{n}[\sqrt{x_1} + \sqrt{x_2} + ... + \sqrt{x_i} + ... + \sqrt{x_n}]$$

Similarly, we can also define the area as $\lim_{n\to\infty} L_n$ or $\lim_{n\to\infty} M_n$, where L_n is the left endpoint approximation and M_n is the midpoint approximation (see page 290 of the text).

We now generalize this process:

To find the area under the curve $y = f(x)$ on $[a, b]$:

Divide $[a, b]$ into n subintervals using partition

$$a = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = b$$

This creates n subintervals:

Then consider n rectangles, one for each subinterval:

Width $\Delta x =$

Height: $f(x_i)$, where x_i is

Area A can be approximated by the sum of the areas of the n rectangles:

This sum is called a **Riemann sum**.

Summation Notation

We use summation notation to write sums in compact form:

$$\sum_{i=m}^{n} a_i =$$

ex. $$\sum_{i=1}^{4} i^3 =$$

ex. $$\sum_{k=2}^{5} (k^2 - 1) =$$

Now, we use summation notation to express the sum more concisely as

$$A \approx$$

Generally, if f is continuous, as the number of subintervals gets larger and widths get smaller the approximation is closer to actual area. We can then define

$$A =$$

ex.

1) Find an expression for the exact area under $f(x) = x^2+1$ from $x = 0$ to $x = 3$ as the limit of a Riemann sum with n subintervals of equal width.

2) Consider the following formula

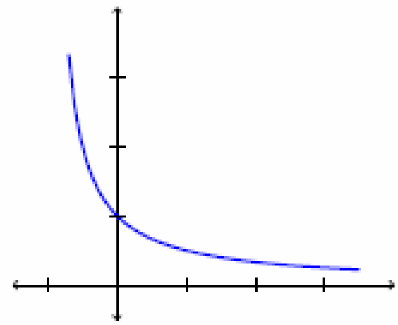
$$1^2 + 2^2 + ... + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use it to find the exact area under $f(x) = x^2+1$ from $x = 0$ to $x = 3$ by evaluating the limit of the Riemann sum:

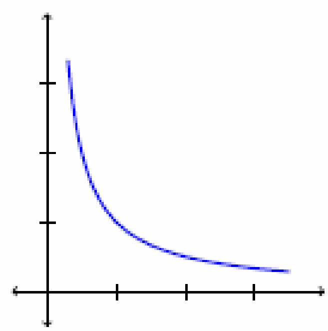
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{3i}{n} \right)^2 + 1 \right) \left(\frac{3}{n} \right)$$

ex. What area is represented by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{1 + \frac{2i}{n}} \right) \left(\frac{2}{n} \right)$$



NOTE:



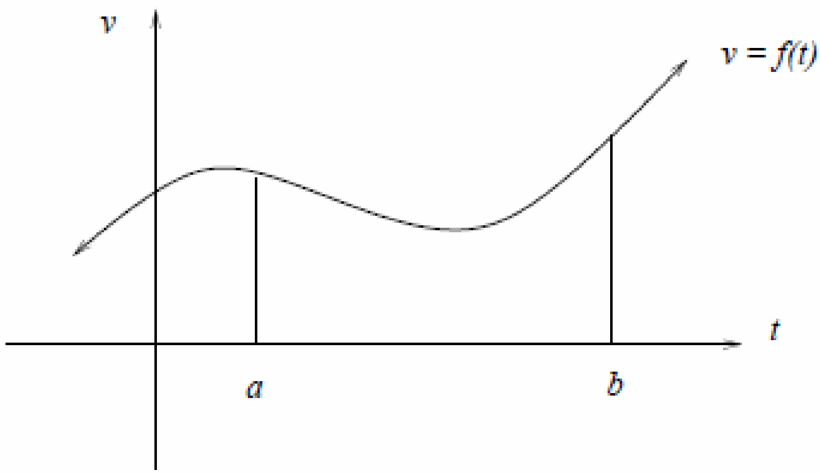
Application of Area: Distance

ex. Suppose an object moves along a track, and its velocity in feet per second is measured every five seconds over a 20 second time interval as recorded in the following table:

Time	0	5	10	15	20
Velocity(ft/sec)	24	30	36	40	45

Estimate the distance traveled over the 20 second interval.

Find the distance traveled by an object during a certain time interval $[a, b]$ if the velocity is known at all times (and is positive).



The Definite Integral

<u>**Def.**</u> If f is defined for $a \leq x \leq b$, divide $[a, b]$ into n subintervals of equal width

$$\Delta x =$$

Let $x_0(= a), x_1, x_2, ..., x_n(= b)$ be the endpoints of these subintervals and let x_i be the right endpoint in the subinterval $[x_{i-1}, x_i]$.

The **definite integral** of f from a to b is

if the limit exists. If so, f is **integrable** on $[a, b]$.

The sum $\sum_{i=1}^{n} f(x_i)\Delta x$ is a

It is used to approximate the definite integral.

Notation

Integral sign

Integrand

Integration

Limits of integration (lower and upper)

dx

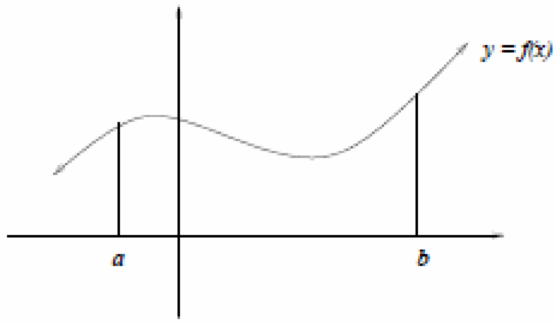
NOTE:

ex. Express $\lim_{n\to\infty}\sum_{i=1}^{n} x_i e^{(x_i)^2-3}\Delta x$ as a definite integral on $[0, 4]$.

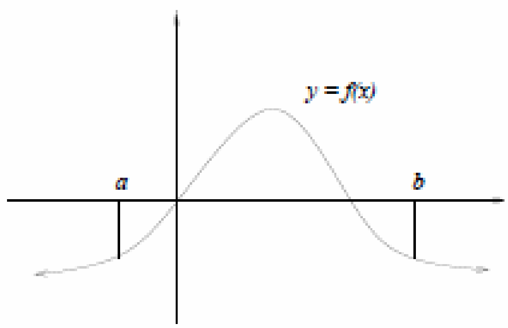
Theorem: If f is continuous or has a finite number of jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$.

Riemann Sums, Definite Integral, and Area:

If $f(x) \geq 0$ on $[a, b]$



If $f(x) \leq 0$ for some x in $[a, b]$



NOTE:

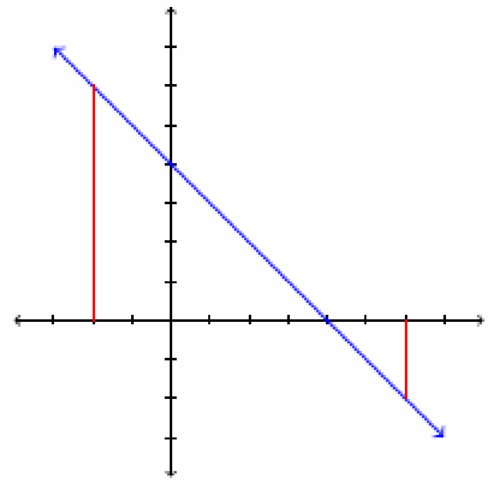
Signed area of a region =

$$\int_a^b f(x)\,dx =$$

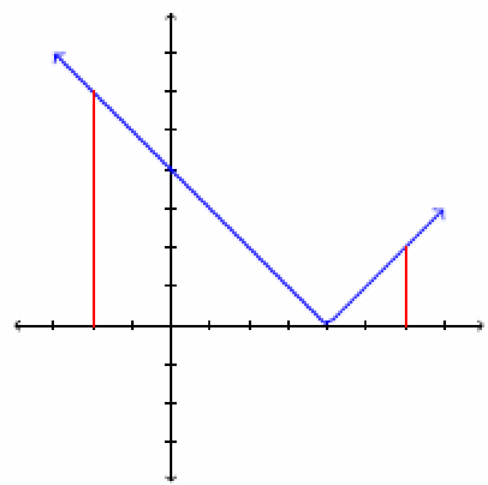
$$\int_a^b |f(x)|\,dx =$$

Evaluating Definite Integrals as Signed Area

ex. Evaluate $\int_{-2}^{6} (4 - x)\, dx$.



ex. Evaluate $\int_{-2}^{6} |4 - x|\, dx$.



To evaluate definite integrals using sums

If c is any constant and if n is a positive integer, then

1. $\sum_{i=1}^{n} ca_i =$

2. $\sum_{i=1}^{n} (a_i + b_i) =$

3. $\sum_{i=1}^{n} (a_i - b_i) =$

4. $\sum_{i=1}^{n} c =$

5. $\sum_{i=1}^{n} i =$

6. $\sum_{i=1}^{n} i^2 =$

7. $\sum_{i=1}^{n} i^3 =$

NOTE: Using right endpoints, if f is integrable on $[a, b]$,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

with $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.

<u>ex.</u> Evaluate $\displaystyle\int_0^3 (x^2 - 3x)\,dx$.

Properties of integrals

1. $\int_b^a f(x)\,dx =$

2. $\int_a^a f(x)\,dx =$

3. If c is a constant, $\int_a^b c\,dx =$

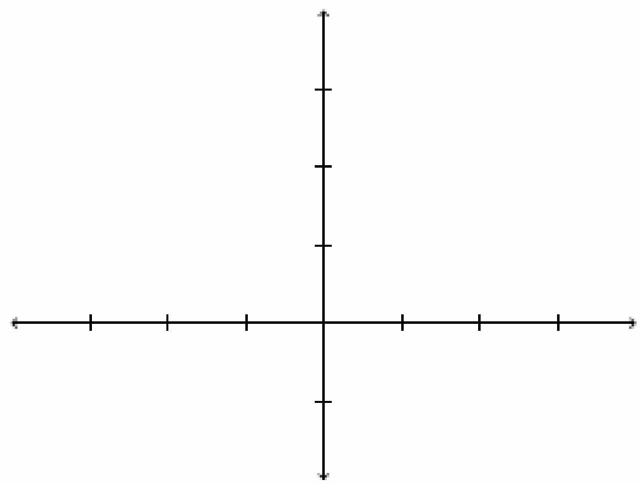
4. $\int_a^b [f(x) \pm g(x)]\,dx =$

5. $\int_a^b cf(x)\,dx =$

6. $\int_a^b f(x)\,dx =$

ex. If $f(x) = \begin{cases} 2 & x < 0 \\ \sqrt{4 - x^2} & x \geq 0 \end{cases}$,

find $\displaystyle\int_{-3}^{2} f(x)\, dx$.



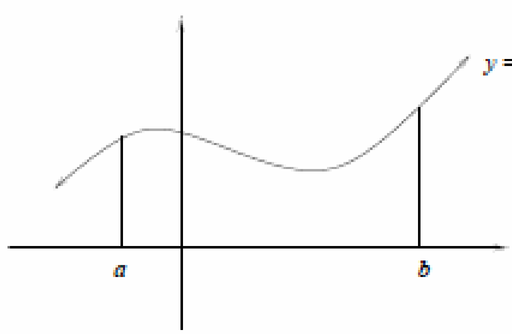
Comparison Properties of Integrals

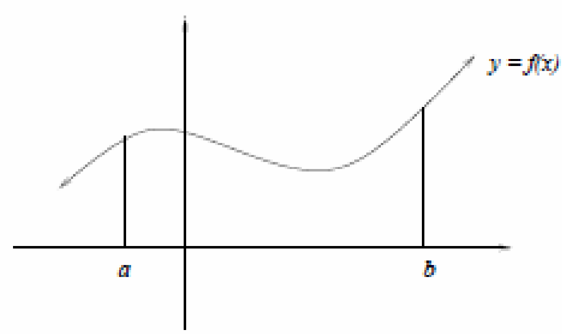
1. If $f(x) \geq 0$ for $a \leq x \leq b$, then

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

We can see this if f is continuous on $[a, b]$:





<u>**ex.**</u> Find the maximum and minimum values of

$$f(x) = \sqrt{x^2 + 1} \text{ on } [-1, 1],$$

and use them to find upper and lower bounds for the value of $\displaystyle\int_{-1}^{1} \sqrt{x^2 + 1}\, dx$.