How to find the area of the region that lies under the curve $y = f(x)$ from $a$ to $b$?

**ex.** Let $f(x) = \sqrt{x}$ and consider the area beneath the graph of the function on $[0, 4]$.

Let $R_n$ be the sum of the areas of $n$ rectangles with equal width and height
1) Find $R_4 \ (n = 4)$:

2) Find $R_8 \ (n = 8)$:
For any $n$,

$$R_n =$$

As $n \to \infty$, what happens to our approximation?

We define the area as $\lim_{n \to \infty} R_n$

$$= \lim_{n \to \infty} \frac{4}{n} \left[ \sqrt{x_1} + \sqrt{x_2} + \ldots + \sqrt{x_i} + \ldots + \sqrt{x_n} \right]$$

Similarly, we can also define the area as $\lim_{n \to \infty} L_n$ or $\lim_{n \to \infty} M_n$, where $L_n$ is the left endpoint approximation and $M_n$ is the midpoint approximation (see page 290 of the text).
We now generalize this process:

**To find the area under the curve** $y = f(x)$ on $[a, b]$:

Divide $[a, b]$ into $n$ subintervals using partition

\[ a = \quad = b \]

This creates $n$ subintervals:

Then consider $n$ rectangles, one for each subinterval:

Width $\Delta x =$

Height: $f(x_i)$, where $x_i$ is

Area $A$ can be approximated by the sum of the areas of the $n$ rectangles:

This sum is called a **Riemann sum**.
Summation Notation

We use summation notation to write sums in compact form:

\[ \sum_{i=m}^{n} a_i = \]

\[ \text{ex. } \sum_{i=1}^{4} i^3 = \]

\[ \text{ex. } \sum_{k=2}^{5} (k^2 - 1) = \]

Now, we use summation notation to express the sum more concisely as

\[ A \approx \]

Generally, if \( f \) is continuous, as the number of subintervals gets larger and widths get smaller the approximation is closer to actual area. We can then define

\[ A = \]
ex.

1) Find an expression for the exact area under \( f(x) = x^2 + 1 \)
from \( x = 0 \) to \( x = 3 \) as the limit of a Riemann sum with \( n \)
subintervals of equal width.
2) Consider the following formula

\[ 1^2 + 2^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}. \]

Use it to find the exact area under \( f(x) = x^2 + 1 \) from \( x = 0 \) to \( x = 3 \) by evaluating the limit of the Riemann sum:

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \left( \left( \frac{3i}{n} \right)^2 + 1 \right) \left( \frac{3}{n} \right) \]
ex. What area is represented by

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{1}{1 + \frac{2i}{n}} \right) \left( \frac{2}{n} \right) \]

NOTE:
Application of Area: Distance

**ex.** Suppose an object moves along a track, and its velocity in feet per second is measured every five seconds over a 20 second time interval as recorded in the following table:

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity (ft/sec)</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td>40</td>
<td>45</td>
</tr>
</tbody>
</table>

Estimate the distance traveled over the 20 second interval.
Find the distance traveled by an object during a certain time interval \([a, b]\) if the velocity is known at all times (and is positive).
The Definite Integral

**Def.** If \( f \) is defined for \( a \leq x \leq b \), divide \([a, b]\) into \( n \) subintervals of equal width

\[
\Delta x =
\]

Let \( x_0(=a), x_1, x_2, ..., x_n(=b) \) be the endpoints of these subintervals and let \( x_i \) be the right endpoint in the subinterval \([x_{i-1}, x_i]\).

The **definite integral** of \( f \) from \( a \) to \( b \) is

if the limit exists. If so, \( f \) is **integrable** on \([a, b]\).

The sum \( \sum_{i=1}^{n} f(x_i)\Delta x \) is a

It is used to approximate the definite integral.
Notation

Integral sign

Integrand

Integration

Limits of integration (lower and upper)

$dx$

NOTE:

ex. Express $\lim_{n \to \infty} \sum_{i=1}^{n} x_i e^{(x_i)^2-3} \Delta x$ as a definite integral on $[0, 4]$.

Theorem: If $f$ is continuous or has a finite number of jump discontinuities on $[a, b]$, then $f$ is integrable on $[a, b]$. 
Riemann Sums, Definite Integral, and Area:

If \( f(x) \geq 0 \) on \([a, b]\)

\[
\begin{align*}
\int_{a}^{b} f(x) \, dx &= \\
\int_{a}^{b} |f(x)| \, dx &=
\end{align*}
\]

NOTE:
Signed area of a region =

\[
\int_{a}^{b} f(x) \, dx =
\]

\[
\int_{a}^{b} |f(x)| \, dx =
\]
Evaluating Definite Integrals as Signed Area

ex. Evaluate $\int_{-2}^{6} (4 - x) \, dx$.

ex. Evaluate $\int_{-2}^{6} |4 - x| \, dx$. 
To evaluate definite integrals using sums

If $c$ is any constant and if $n$ is a positive integer, then

1. $\sum_{i=1}^{n} ca_i =$

2. $\sum_{i=1}^{n} (a_i + b_i) =$

3. $\sum_{i=1}^{n} (a_i - b_i) =$

4. $\sum_{i=1}^{n} c =$

5. $\sum_{i=1}^{n} i =$

6. $\sum_{i=1}^{n} i^2 =$

7. $\sum_{i=1}^{n} i^3 =$
NOTE: Using right endpoints, if $f$ is integrable on $[a, b],
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]
with $\Delta x = \frac{b - a}{n}$ and $x_i = a + i \Delta x$.

ex. Evaluate $\int_0^3 (x^2 - 3x) \, dx$. 
Properties of integrals

1. \( \int_b^a f(x) \, dx = \)

2. \( \int_a^a f(x) \, dx = \)

3. If \( c \) is a constant, \( \int_a^b c \, dx = \)

4. \( \int_a^b [f(x) \pm g(x)] \, dx = \)

5. \( \int_a^b cf(x) \, dx = \)

6. \( \int_a^b f(x) \, dx = \)
ex. If \( f(x) = \begin{cases} 
\frac{2}{\sqrt{4 - x^2}} & x < 0 \\
\sqrt{4 - x^2} & x \geq 0 
\end{cases} \)

find \( \int_{-3}^{2} f(x) \, dx \).
Comparison Properties of Integrals

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

We can see this if $f$ is continuous on $[a, b]$:
Ex. Find the maximum and minimum values of 

\[ f(x) = \sqrt{x^2 + 1} \text{ on } [-1, 1], \]

and use them to find upper and lower bounds for the value of 

\( \int_{-1}^{1} \sqrt{x^2 + 1} \, dx. \)