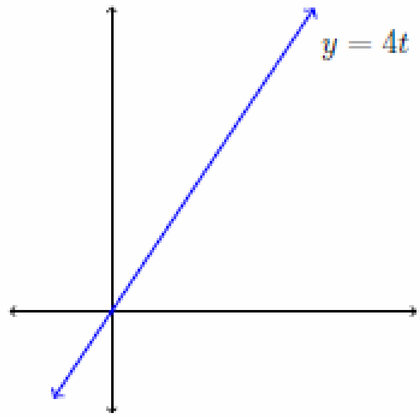
L16 The Fundamental Theorem of Calculus

ex. Given the function $y = f(t) = 4t$:



Now consider the function of x for $x \geq 0$ defined by

$$A(x) = \int_0^x 4t\, dt.$$

For any given x, we know that $A(x)$ gives the area of the region bounded by the graph of $f(t)$ from $t = 0$ to $t = x$.

1) Find an expression for $A(x)$.

2) Find $A'(x)$.

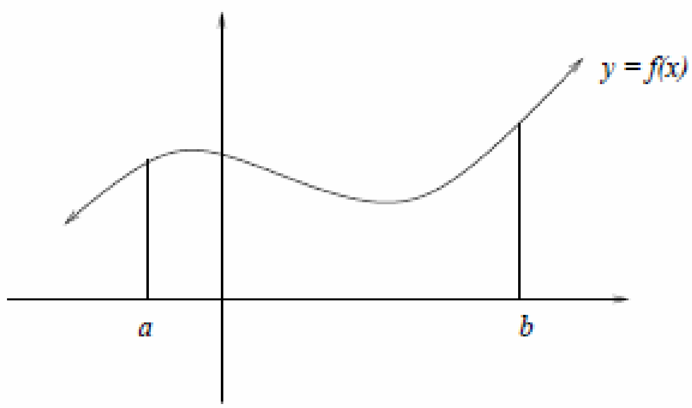
The Fundamental Theorem of Calculus, Part I:

If f is continuous on $[a, b]$, then the function A defined by

$$A(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b) and

$$A'(x) =$$



NOTE: If f is continuous, $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt =$

ex. If $g(x) = \int_1^x t\cos(t^2)\,dt$, find $g'(x)$.

ex. Find $h'(x)$ if $h(x) = \int_2^x \frac{t^2+1}{t}\,dt$.

The FTC and the Chain Rule

ex. Find $\frac{dF}{dx}$ if $F(x) = \int_0^{x^2} t\sqrt{t^3+4}\,dt$.

ex. Find $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^{\tan x} e^{2t}\,dt$.

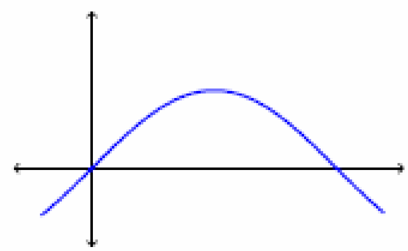
Fundamental Theorem of Calculus, Part II:

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx =$$

where F is any antiderivative of f. (That is, $F'(x) =$)

ex. Evaluate $\int_0^{\frac{\pi}{2}} \sin x \, dx$ and sketch the area represented by this definite integral.



ex. Evaluate $\int_0^{\ln 2} e^{3x} \, dx$.

ex. 1) Evaluate $\int_1^4 (x - \sqrt{x})^2 \, dx$.

2) Evaluate $\int_4^4 (x - \sqrt{x})^2 \, dx$.

ex. Find the area bounded by $f(x) = e^x + \dfrac{2}{x}$ and the x-axis from $x = 1$ to $x = e$.

Now consider the area bounded by $f(x)$ and the x-axis from $x = 0$ to $x = e$.

ex. Find the area of the region bounded by the x-axis and the graph of $f(x) = \begin{cases} x+1 & x \leq 0 \\ \sec^2 x & x > 0 \end{cases}$ on the interval $\left[-1, \frac{\pi}{4}\right]$.

ex. For a given value of x, we can use the FTC, part II to find

$$h(x) = \int_2^x \frac{t^2 + 1}{t}\, dt.$$

Now find $h'(x)$. Compare this with the result found previously.

To summarize the relationship between the parts of

The Fundamental Theorem of Calculus

If f is continuous on $[a, b]$,

1) If $A(x) = \int_a^x f(t)\,dt$, then

2) $\int_a^b f(x)\,dx =$

Note the following:

ex. $\dfrac{d}{dx} \displaystyle\int_a^x (t^3 + 2t)\,dt$

ex. $\displaystyle\int_0^x \dfrac{d}{dt}(\sec t)\,dt$

Indefinite Integrals & Net Change Theorem

Note the connections between antiderivatives and the definite integral from:

Fundamental Theorem of Calculus, part I

If f is continuous, then $\int_a^x f(t)\,dt$ is

Fundamental Theorem of Calculus, part II

$$\int_a^b f(x)\,dx =$$

where

Indefinite Integrals

Definite Integrals

We rewrite our antiderivative formulas as integrals:

1. $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$

2. $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

3. $\displaystyle\int k\,dx =$

4. $\displaystyle\int x^n\,dx =$

5. $\displaystyle\int \frac{1}{x}\,dx =$

6. $\displaystyle\int e^x\,dx =$

7. $\displaystyle\int a^x\,dx =$

8. $\displaystyle\int \cos x\,dx =$

9. $\displaystyle\int \sin x\,dx =$

10. $\int \sec^2 x \, dx =$

11. $\int \csc^2 x \, dx =$

12. $\int \sec x \tan x \, dx =$

13. $\int \csc x \cot x \, dx =$

14. $\int \frac{1}{\sqrt{1-x^2}} \, dx =$

15. $\int \frac{1}{1+x^2} \, dx =$

ex. $\int \left(2^x + \frac{4}{\sqrt{1-x^2}} - e^{2x}\right) dx$

ex. $$\int \frac{\sqrt[4]{x^3}+1}{x}\,dx$$

ex. $$\int \frac{x^2}{x^2+1}\,dx$$

ex. $\int \tan^2 x \, dx$

ex. $\int_0^{\frac{\pi}{4}} \frac{\sin \theta}{\cos^2 \theta} \, d\theta$

Net Change Theorem

The integral of a rate of change of a function is the **net change** of the function on the interval $[a, b]$:

$$\int_a^b F'(x)\,dx =$$

<u>**ex.**</u> If the volume of water in a lake is increasing at the rate $V'(t)$, then

$$\int_{t_1}^{t_2} V'(t)\,dt =$$

gives

<u>**ex.**</u> If a population is growing at a rate of $\dfrac{dn}{dt}$, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt =$$

gives

ex. Suppose that a population of birds is increasing at the rate of $100 + 20t$ birds per year. What is the net increase in population between the sixth and eight years?

Suppose a particle is moving along a straight line with position function $s(t)$, velocity $v(t)$, and acceleration $a(t)$. Then

$$\int_{t_1}^{t_2} v(t)\, dt =$$

$$\int_{t_1}^{t_2} a(t)\, dt =$$

NOTE: total distance traveled =

ex. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$ m/sec.

1) Find the displacement during the time $2 \leq t \leq 5$.

2) Find the total distance traveled during the time $2 \leq t \leq 5$.