L2. Limits, Velocity and Tangent Lines

Average velocity =

Velocity has both direction and magnitude. speed= |velocity|.

ex. Suppose that as an object falls from the top of a cliff, its position in feet above the ground after t seconds is given by the formula $s(t) = 160 - 16t^2$.

1) Find the average velocity of the object on the time interval from $t = 1$ to $t = 3$.

2) Find the average velocity on the time interval $t = 1$ to $t = 2$.

Consider the following table for our position function:

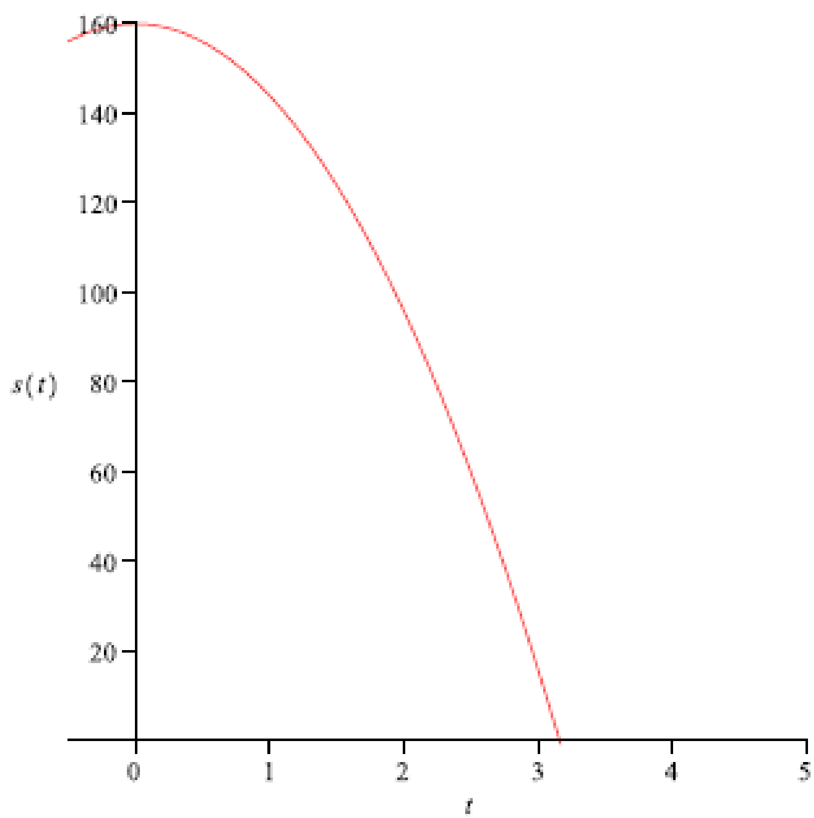
Time Interval	Average Velocity
[1, 2]	-48 ft/sec
[1, 1.5]	-40 ft/sec
[1, 1.1]	-33.6 ft/sec
[1, 1.01]	-32.16 ft/sec
[1, 1.001]	-32.016 ft/sec

Can we find the rate at which the object is falling at **exactly** one second?

We generalize the above process: for a small number h, find the average velocity of $s(t) = 160 - 16t^2$ from $t = 1$ to $t = 1 + h$ seconds.

Now we can find the **instantaneous velocity** of the object when $t = 1$.

To see this from another point of view, consider the graph of $s(t) = 160 - 16t^2$:



A **secant line** is a line joining two points on a curve. On our example, consider the slope of the secant line through

1) $[1, 3]$

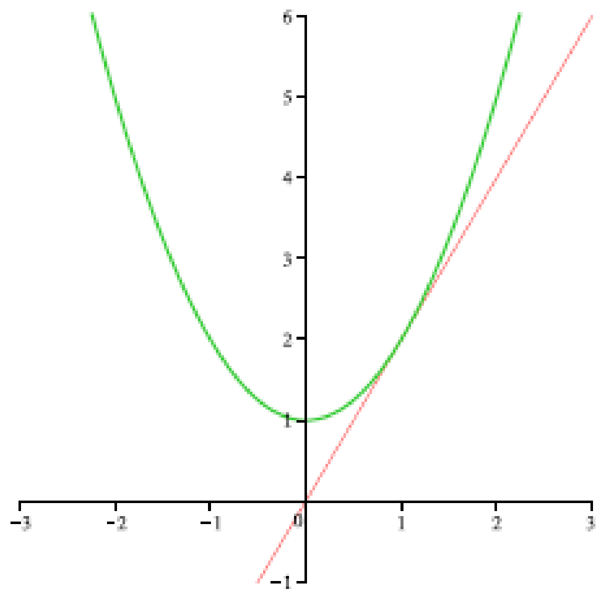
2) $[1, 2]$

We observe: Slope of secant line

Notice that the secant lines in our above graph approach a unique line, called the **tangent line** of the graph at $t = 1$.

Slope of the tangent line

ex. Find the slope of the tangent to $y = x^2 + 1$ at $(1, 2)$.



Consider points $P = (1, 2)$ and $Q = (x, x^2 + 1)$, $x \neq 1$

slope of secant line through P and Q:

$m_{sec} =$

How can we find the slope of the tangent line at $(1, 2)$?

If $f(x) = m_{sec} = \dfrac{x^2-1}{x-1}$, consider the following:

x	$f(x)$
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

x	$f(x)$
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

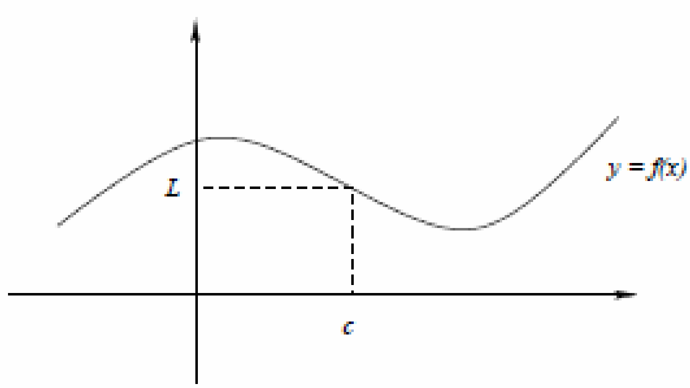
Note that $f(x)$ as x

so in our example, slope of the tangent line $= m_{tan} =$

We use the notation: $\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{x^2-1}{x-1} = 2$

Introduction to Limits

Def. Let $f(x)$ be a function. We say that $\lim_{x\to c} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to c (on either side of c) but not equal to c.



What do we mean by making $f(x)$ arbitrarily close to L?

Consider our function $f(x) = \dfrac{x^2 - 1}{x - 1}$, $x \neq 1$:

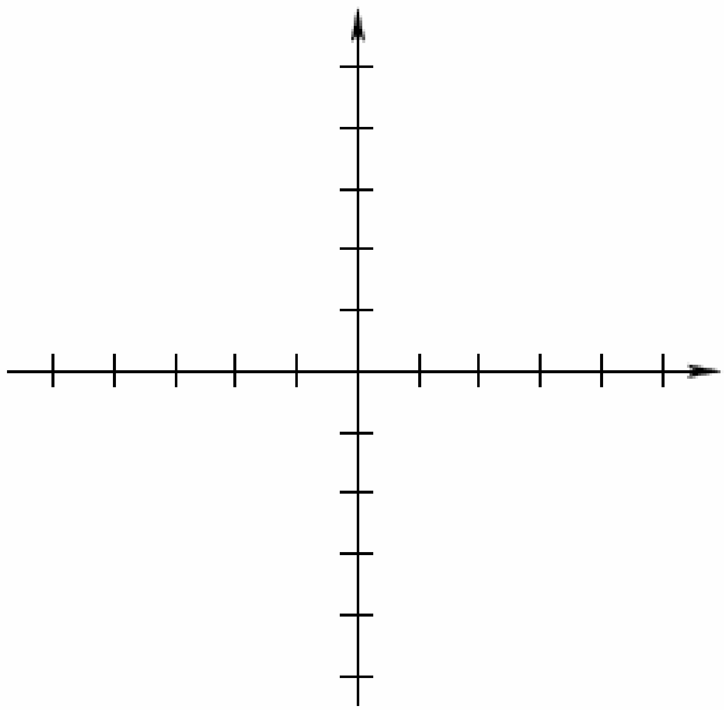
We write $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} =$

To show that $\lim_{x \to 1} f(x) = \lim_{x \to 1} x + 1 = 2$, we must show that $|f(x) - 2|$ becomes arbitrary small when x is sufficiently close to 1 (but $x \neq 1$):

$$|f(x) - 2| =$$

We can also see these ideas graphically:

Sketch the graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$.

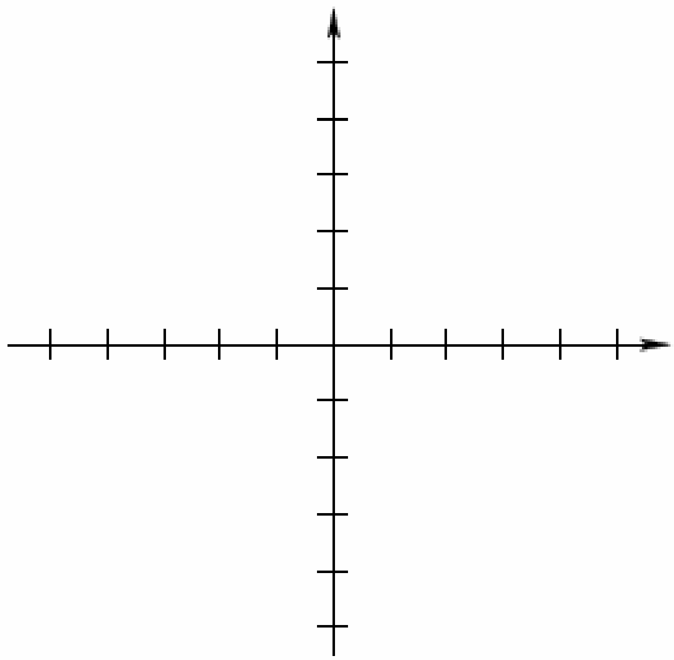


a) What is the domain of f?

b) $\lim_{x \to 1} f(x) =$

Now consider $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & x \neq 1 \\ 3 & x = 1 \end{cases}$

Sketch the graph of $g(x)$.



a) What is the domain of g?

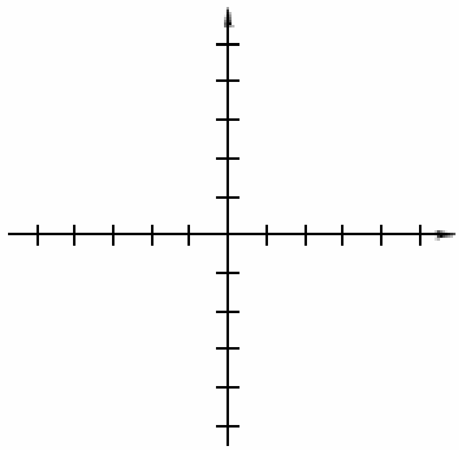
b) $\lim_{x \to 1} g(x) =$

NOTE: $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) =$

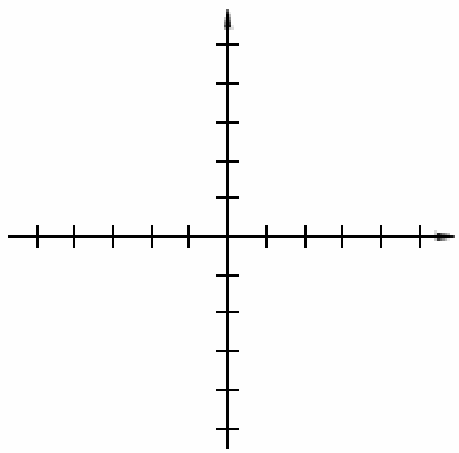
but $f(1) =$ and $g(1) =$

Recall the definition: For a given function $f(x)$, we say that $\lim_{x \to c} f(x) = L$ if we can make the values of $f(x)$ as close to L as we want by choosing x sufficiently close to c on either side but not equal to c.

<u>ex.</u> If $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$, find $\lim_{x \to 1} f(x)$.



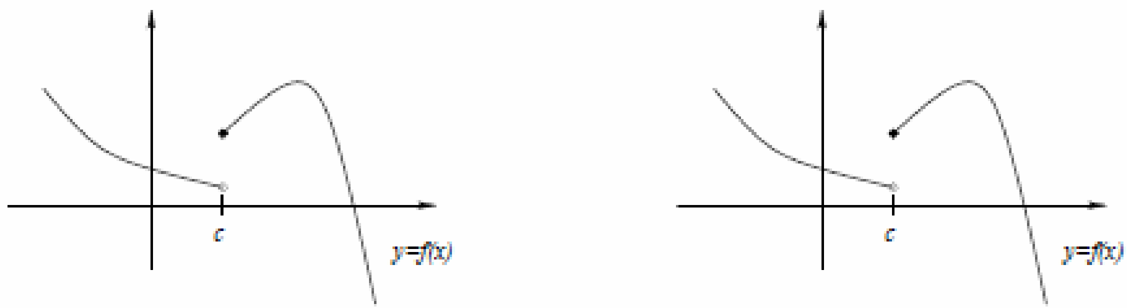
<u>ex.</u> If $g(x) = \begin{cases} 3 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0 \end{cases}$, find $\lim_{x \to 0} g(x)$.



Def. One-Sided Limits

We say that a function $f(x)$ has limit L as x approaches the number c from the right if we can make every value of $f(x)$ as close to L as we want by choosing x sufficiently close to c but $x > c$.

We write this **right-hand limit** as $\lim_{x\to c^+} f(x) = L$.



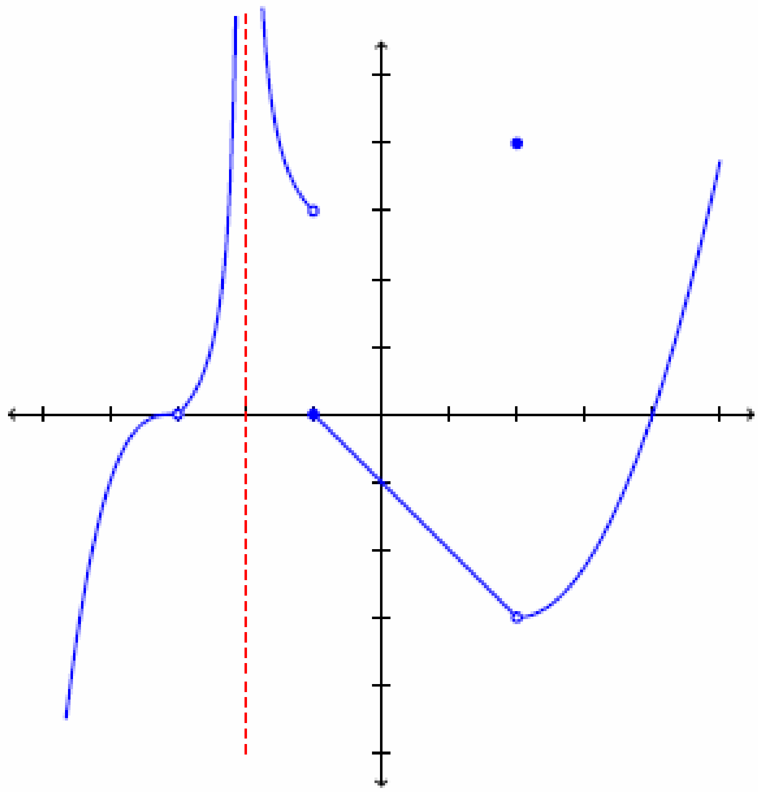
Similarly, the **left-hand limit** $\lim_{x\to c^-} f(x) = L$ means we can make every value of $f(x)$ as close to L as we want by choosing x sufficiently close to c but $x < c$.

In the example $g(x) = \begin{cases} 3 & x \leq 0 \\ -1 & x > 0 \end{cases}$

$\lim_{x\to 0^-} g(x) =$ $\qquad\qquad$ $\lim_{x\to 0^+} g(x) =$

NOTE: $\lim_{x\to c} f(x) = L$ if and only if

ex. From the given graph, find the limits:



$\lim_{x \to -3} f(x) =$

$\lim_{x \to 2^-} f(x) =$

$\lim_{x \to -1^+} f(x) =$

$\lim_{x \to 2^+} f(x) =$

$\lim_{x \to -1^-} f(x) =$

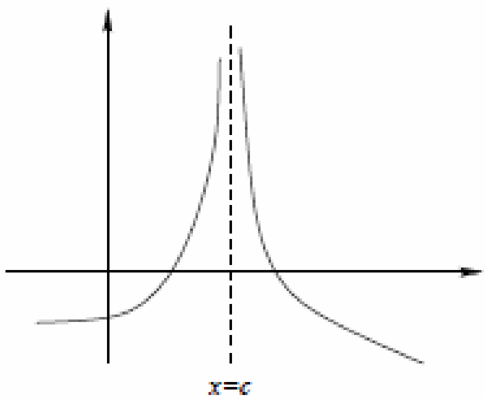
$\lim_{x \to 2} f(x) =$

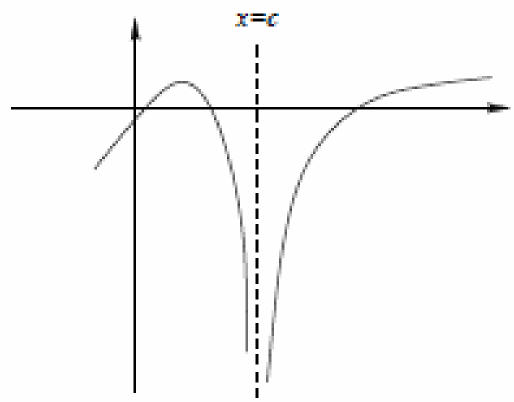
$\lim_{x \to -1} f(x) =$

$\lim_{x \to -2} f(x) =$

Infinite Limits

Def. Let f be defined on both sides of c, except possibly at c. $\lim_{x \to c} f(x) = \infty$ if the values of $f(x)$ can be made (and kept) as large as we want by taking x sufficiently close to c but not equal to c.





Also, $\lim_{x \to c} f(x) = -\infty$ means the values $f(x) < 0$ can be made (and kept) as large as possible in absolute value as we want by taking x sufficiently close but not equal to c.

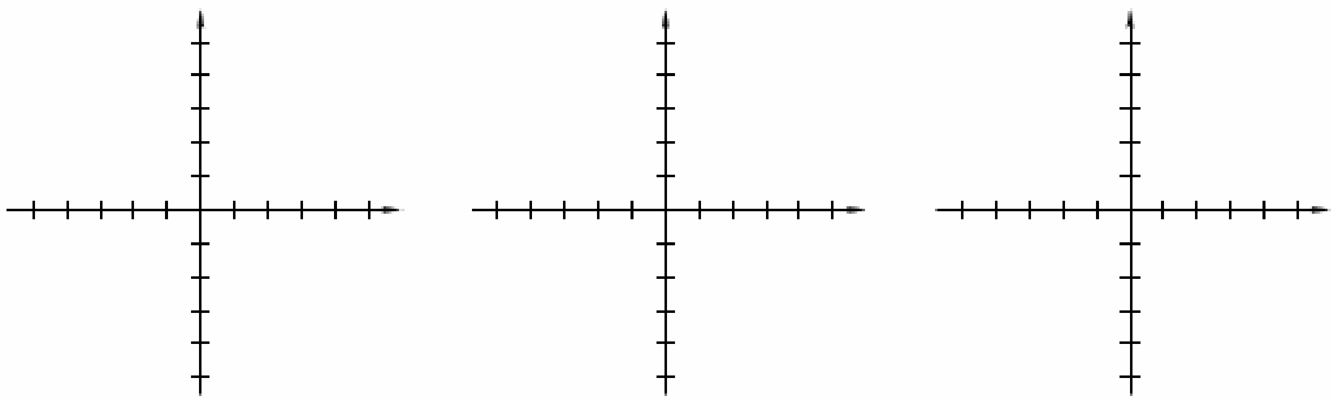
NOTE: Similar definitions can be given for approaching c from the left or right.

Def. The line $x = c$ is called a

of the curve $y = f(x)$ if at least one of the following is true:

ex. Graph the given functions:

$$f(x) = \frac{1}{x-1} \qquad g(x) = \frac{1}{(x-1)^2} \qquad h(x) = -\frac{1}{(x-1)^2}$$



ex. Use the graphs to find the following limits:

1) $\lim_{x \to 1^-} f(x) =$

$\lim_{x \to 1^+} f(x) =$

$\lim_{x \to 1} f(x) =$

2) $\lim_{x \to 1} g(x) =$

3) $\lim_{x \to 1} h(x) =$

Note the form of $f(1)$, $g(1)$ and $h(1)$ for the above examples, and their relation to the limit of each function as $x \to 1$.

<u>**ex.**</u> Use the following table of values to evaluate

$$\lim_{x \to 1^-} \frac{x-5}{x-1}.$$

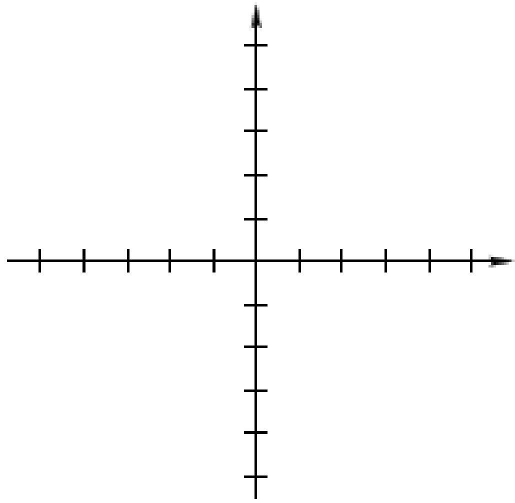
x	$\frac{x-5}{x-1}$
0.9	41
0.99	401
0.999	4,001
0.9999	40,001

<u>**ex.**</u> Find each vertical asymptote of $f(x) = \dfrac{x-5}{x-1}$.

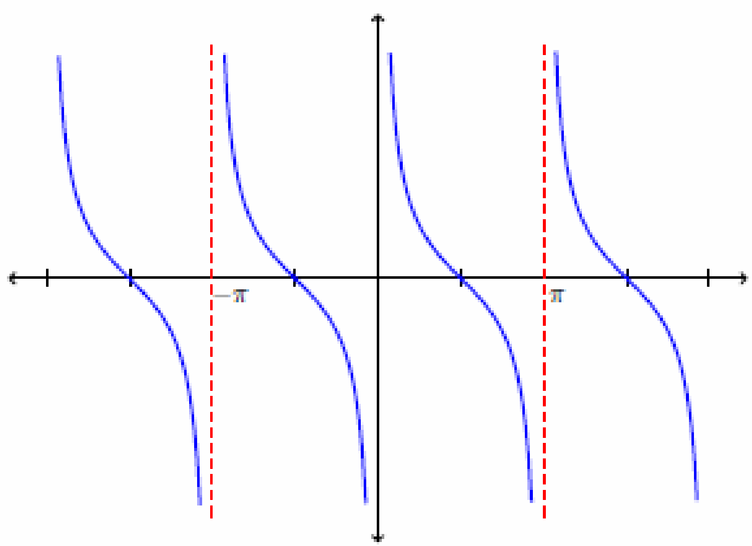
<u>**ex.**</u> Find each vertical asymptote of $f(x) = \dfrac{x^2-1}{x-1}$

ex. Evaluate $\lim_{x \to 1^+} \ln(x - 1)$.

Find any vertical asymptotes of $f(x) = \ln(x - 1)$.



ex. Find $\lim_{x \to \pi^-} \cot x$.



Find any vertical asymptotes of $g(x) = \cot x$.