L3. Evaluating Limits, Continuity

Basic Limit Laws

Suppose \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) exist and \( k \) is a constant. We have the following limit laws:

1. \( \lim_{x \to c} [ f(x) \pm g(x) ] = \)

2. \( \lim_{x \to c} k f(x) = \)

3. \( \lim_{x \to c} f(x) g(x) = \)

4. If \( \lim_{x \to c} g(x) \neq 0 \), \( \lim_{x \to c} \frac{f(x)}{g(x)} = \)

5. If \( p, q \) are integers, with \( q \neq 0 \), then \( \lim_{x \to c} [ f(x) ]^{p/q} = \)

Assume that \( \lim_{x \to c} f(x) \geq 0 \) if \( q \) is even, and that \( \lim_{x \to c} f(x) \neq 0 \) if \( p/q < 0 \).

In particular, for a positive integer \( n \):

\( \lim_{x \to c} [ f(x) ]^n = \), \( \lim_{x \to c} n \sqrt[n]{f(x)} = \)
Note the following (Theorem 1, Section 2.2):

6. \( \lim_{x \to c} k = \)

7. \( \lim_{x \to c} x = \)

**ex.** Evaluate \( \lim_{x \to -1} (2x^2 + 3x - 2) \).

Note that \( \lim_{x \to -1} f(x) = \lim_{x \to -1} (2x^2 + 3x - 2) = f(-1) \). That is, we can find the limit by direct substitution.

**Direct Substitution Property**

If \( f(x) \) is a polynomial and \( c \) is in the domain of \( f \) then

\[ \lim_{x \to c} f(x) = \]
What about a rational function $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials? If $q(c) \neq 0$, we can extend the substitution property:

$$\lim_{x \to c} R(x) = \lim_{x \to c} \frac{p(x)}{q(x)} =$$

**ex.** Evaluate $\lim_{x \to 1} \frac{x^2 - 5}{x + 1}$.

**ex.** Evaluate $\lim_{x \to -3} \frac{x^2 + 2x - 3}{x + 3}$.
We are using an important result:

If $f$ and $g$ are functions so that $f(x) = g(x)$ for all $x \neq c$, and $\lim_{x \to c} g(x)$ exists, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$.

So if $h(x) = \frac{f(x)}{g(x)}$ and both $f(x)$ and $g(x)$ approach 0 as $x \to c$, the indeterminate form $\frac{0}{0}$, we use algebraic techniques to find a function equivalent to $h(x)$ except at $x = c$.

**ex.** Evaluate: $\lim_{x \to 3} \frac{x - \sqrt{12 - x}}{x - 3}$
ex. \[ \lim_{x \to 4} \frac{\frac{1}{x} - \frac{4}{x^2}}{4 - x} \]

Now let \( f(x) = \begin{cases} \frac{1}{x} - \frac{4}{x^2} & x \neq 4, \\ \frac{x}{4 - x} & x = 4 \end{cases} \)

What is \( \lim_{x \to 4} f(x) \)?
Indeterminate Forms: \( \frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty, 1^\infty, \infty^0, 0^0 \)

(We will discuss indeterminate powers in Section 4.5.)

**ex.** Evaluate \( \lim_{x \to 1} \frac{1}{\sqrt{x - 1}} - \frac{2}{x - 1} \)

The following quotients are **not** indeterminate forms:

**ex.** Evaluate \( \lim_{x \to 0^+} \frac{x + 2}{\ln x} \)

**ex.** Evaluate \( \lim_{x \to 2^-} \frac{x - 4}{x - 2} \)
Recall One-Sided Limits

Theorem. \( \lim_{x \to c} f(x) = L \) if and only if

\[\text{ex. Given } g(x) = \begin{cases} 
-2 & x \leq -2 \\
4 - x^2 & -2 < x < 2 \\
\ln(x - 1) & x > 2 
\end{cases}\]

Find the following:

1) \( \lim_{x \to -2} g(x) \)

2) \( \lim_{x \to 2} g(x) \)
3) Sketch the graph of \( g(x) = \begin{cases} 
-2 & x \leq -2 \\
4 - x^2 & -2 < x < 2 \\
\ln(x - 1) & x > 2 
\end{cases} \)
and compare.
Recall: $\sqrt{x^2} =$

**ex.** Find the following:

a) $\lim_{x \to 0^+} x \sqrt{1 + \frac{1}{x^2}}$

b) $\lim_{x \to 0^-} x \sqrt{1 + \frac{1}{x^2}}$

c) $\lim_{x \to 0} x \sqrt{1 + \frac{1}{x^2}}$
**Def.** A function \( f \) is **continuous** at a number \( c \) if

\[
\lim_{{x \to c}} f(x) =
\]

If \( f \) is defined on an open interval including \( x = c \) but is not continuous there, then \( f \) is **discontinuous** at \( c \).

The definition implies three conditions for continuity:

1. 

2. 

3. 

Note that \( f \) is continuous at \( x = c \) if the limit as \( x \to c \) gives the same number as evaluating the function at \( x = c \).
ex. Consider the graph of a function $f(x)$

At which numbers is $f$ discontinuous?

Can we define or redefine $f(x)$ to make it continuous at any of those values?
We classify two types of discontinuities at a point $x = c$:

Removable

Nonremovable

Jump Discontinuity

Infinite Discontinuity
Functions that are continuous

The following familiar functions are continuous for each $x$ in their domain:

Polynomials    Rational functions    Root functions
Trigonometric and Inverse Trigonometric functions
Exponential functions    Logarithmic functions

Theorem. (Basic Laws of Continuity)
If functions $f$ and $g$ are continuous at $c$ and $k$ is a constant, then the following functions are also continuous at $x = c$:

$$f \pm g, \; f \cdot g, \; kf \; \text{and} \; \frac{f}{g} \; \text{if} \; g(c) \neq 0$$

This can be verified by the Basic Limit Laws.

Theorem. (Continuity of Composite Functions)
If $g$ is continuous at $x = c$ and $f$ is continuous at $g(c)$, then the composition $f \circ g$ is continuous at $x = c$. 
ex. On what interval(s) is $f(x) = \sqrt{x + 1} + \frac{\ln x}{x - 4}$ continuous?

ex. Evaluate based on continuity of the function:

$$\lim_{x \to 1} 2^{x-2\sqrt{x}}$$

So now we can use direct substitution to find:

**ex.** $\lim_{x \to \frac{\pi}{2}} \sin(x - \cos x)$
Functions with Discontinuities

ex. Consider \( f(x) = \frac{x - x^2}{x^3 + x^2 - 2x} \). What types of discontinuities does \( f(x) \) have? Could we define \( f(x) \) to make it continuous at any of those discontinuities?
ex. Find any removable or nonremovable discontinuities of

\[ f(x) = \begin{cases} 
  x - 1 & \text{if } x < 0 \\
  |x| + 1 & \text{if } 0 \leq x < 1 \\
  2\sqrt{x} & \text{if } x > 1 
\end{cases} \]
**Def.** A function $f$ is **continuous from the right** at $x = c$ if

$$f$$ is **continuous from the left** at $c$ if

In our example, is $f$ continuous from the right or left at $x = 0$?

At $x = 1$?

**Continuity on Intervals**

A function is continuous on an interval $I$ if it is continuous at each $x$-value in $I$. If $f$ is defined only on one side of an endpoint of the interval, we consider continuity to be continuity from the left or right.

On an interval for which a function is continuous its graph has no jumps, holes, or gaps.
**ex.** Find the value of $K$ which will make $f(x)$ continuous at $x = e$.

$$f(x) = \begin{cases} 
  \ln(2x - e) & x \leq e \\
  4x - K & x > e
\end{cases}$$
An important property of continuous functions is the Intermediate Value Theorem (IVT):

Suppose $f$ is continuous on $[a, b]$ and $M$ is any number between $f(a)$ and $f(b)$, then

![Graph of a function](image)

**Corollary. (Existence of Zeros)**

If $f(x)$ is continuous on $[a, b]$ and if $f(a) > 0$ and $f(b) < 0$ (or $f(a) < 0$ and $f(b) > 0$), then $f(x)$ has a zero in $(a, b)$.

**ex.** Does the IVT imply that $f(x) = \frac{1}{x}$ has a zero on $(-1, 1)$?

**ex.** Use the IVT to show that the equation $x^3 + 3x = 2$ has a root in the interval $(-1, 1)$. 