L3. Evaluating Limits, Continuity

Basic Limit Laws

Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist and k is a constant. We have the following limit laws:

1. $\lim_{x\to c} [\, f(x) \pm g(x) \,] =$

2. $\lim_{x\to c} k\, f(x) =$

3. $\lim_{x\to c} f(x)\, g(x) =$

4. If $\lim_{x\to c} g(x) \neq 0$, $\lim_{x\to c} \dfrac{f(x)}{g(x)} =$

5. If $p,\ q$ are integers, with $q \neq 0$, then $\lim_{x\to c} [\, f(x) \,]^{p/q} =$

Assume that $\lim_{x\to c} f(x) \geq 0$ if q is even, and that $\lim_{x\to c} f(x) \neq 0$ if $p/q < 0$.

In particular, for a positive integer n:

$\lim_{x\to c} [\, f(x) \,]^n =$, $\lim_{x\to c} \sqrt[n]{f(x)} =$

Note the following (Theorem 1, Section 2.2):

6. $\lim_{x \to c} k =$

7. $\lim_{x \to c} x =$

ex. Evaluate $\lim_{x \to -1} (2x^2 + 3x - 2)$.

Note that $\lim_{x \to -1} f(x) = \lim_{x \to -1} (2x^2 + 3x - 2) = f(-1)$. That is, we can find the limit by direct substitution.

Direct Substitution Property

If $f(x)$ is a polynomial and c is in the domain of f then

$$\lim_{x \to c} f(x) =$$

What about a rational function $R(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials? If $q(c) \neq 0$, we can extend the substitution property:

$$\lim_{x \to c} R(x) = \lim_{x \to c} \frac{p(x)}{q(x)} =$$

ex. Evaluate $\displaystyle\lim_{x \to 1} \frac{x^2 - 5}{x + 1}$.

ex. Evaluate $\displaystyle\lim_{x \to -3} \frac{x^2 + 2x - 3}{x + 3}$.

We are using an important result:

If f and g are functions so that $f(x) = g(x)$ for all $x \neq c$, and $\lim_{x \to c} g(x)$ exists, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$.

So if $h(x) = \dfrac{f(x)}{g(x)}$ and both $f(x)$ and $g(x)$ approach 0 as $x \to c$, **the indeterminate form** $\dfrac{0}{0}$, we use algebraic techniques to find a function equivalent to $h(x)$ except at $x = c$.

<u>ex.</u> Evaluate: $\displaystyle \lim_{x \to 3} \frac{x - \sqrt{12 - x}}{x - 3}$

ex. $\lim_{x\to 4} \dfrac{\frac{1}{x} - \frac{4}{x^2}}{4 - x}$

Now let $f(x) = \begin{cases} \dfrac{\frac{1}{x} - \frac{4}{x^2}}{4 - x} & x \neq 4. \\ 2 & x = 4 \end{cases}$

What is $\lim_{x\to 4} f(x)$?

Indeterminate Forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, 1^∞, ∞^0, 0^0

(We will discuss indeterminate powers in Section 4.5.)

ex. Evaluate $\lim_{x \to 1} \frac{1}{\sqrt{x} - 1} - \frac{2}{x - 1}$

The following quotients are **not** indeterminate forms:

ex. Evaluate $\lim_{x \to 0^+} \frac{x + 2}{\ln x}$

ex. Evaluate $\lim_{x \to 2^-} \frac{x - 4}{x - 2}$

Recall One-Sided Limits

Theorem. $\lim_{x \to c} f(x) = L$ if and only if

ex. Given $g(x) = \begin{cases} -2 & x \leq -2 \\ 4 - x^2 & -2 < x < 2 \\ \ln(x-1) & x > 2 \end{cases}$

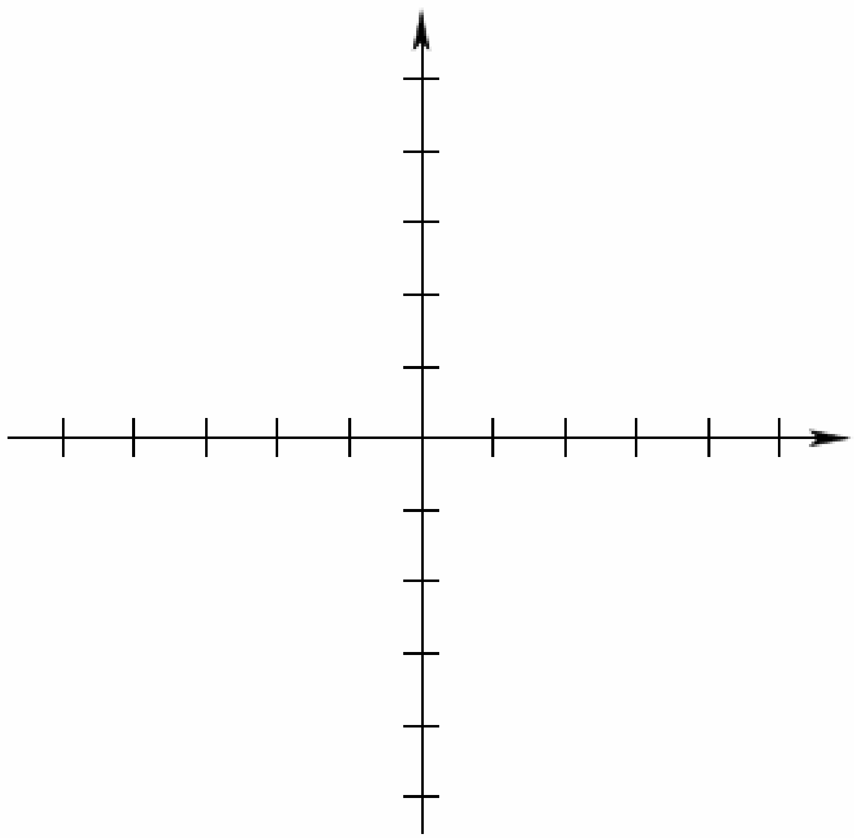
Find the following:

1) $\lim_{x \to -2} g(x)$

2) $\lim_{x \to 2} g(x)$

3) Sketch the graph of $g(x) = \begin{cases} -2 & x \leq -2 \\ 4 - x^2 & -2 < x < 2 \\ \ln(x-1) & x > 2 \end{cases}$

and compare.



Recall: $\sqrt{x^2} =$

ex. Find the following:

a) $\lim_{x \to 0^+} x\sqrt{1 + \frac{1}{x^2}}$

b) $\lim_{x \to 0^-} x\sqrt{1 + \frac{1}{x^2}}$

c) $\lim_{x \to 0} x\sqrt{1 + \frac{1}{x^2}}$

Def. A function f is **continuous** at a number c if

$$\lim_{x \to c} f(x) =$$

If f is defined on an open interval including $x = c$ but is not continuous there, then f is **discontinuous** at c.

The definition implies three conditions for continuity:

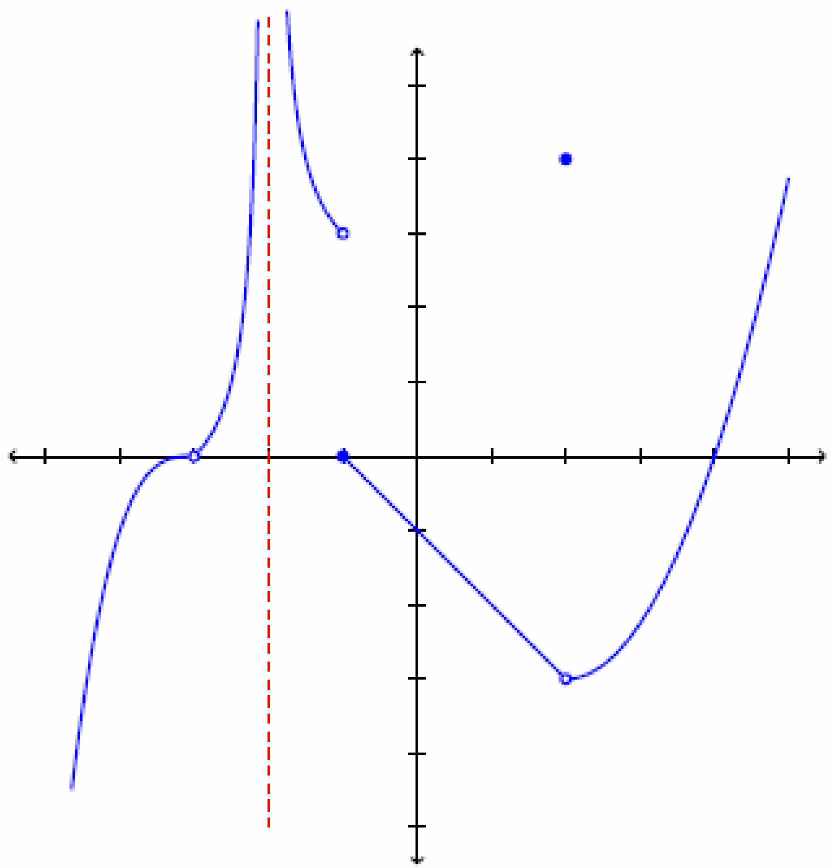
1.

2.

3.

Note that f is continuous at $x = c$ if the limit as $x \to c$ gives the same number as evaluating the function at $x = c$.

ex. Consider the graph of a function $f(x)$



At which numbers is f discontinuous?

Can we define or redefine $f(x)$ to make it continuous at any of those values?

We classify two types of discontinuities at a point $x = c$:

Removable

Nonremovable

Jump Discontinuity

Infinite Discontinuity

Functions that are continuous

The following familiar functions are **continuous for each x in their domain:**

Polynomials Rational functions Root functions

Trigonometric and Inverse Trigonometric functions

Exponential functions Logarithmic functions

Theorem. **(Basic Laws of Continuity)**
If functions f and g are continuous at c and k is a constant, then the following functions are also continuous at $x = c$:

$$f \pm g \ , \ f \cdot g \ , \ kf \ \text{ and } \ \frac{f}{g} \text{ if } g(c) \neq 0$$

This can be verified by the Basic Limit Laws.

Theorem. **(Continuity of Composite Functions)**
If g is continuous at $x = c$ and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at $x = c$.

ex. On what interval(s) is $f(x) = \sqrt{x+1} + \dfrac{\ln x}{x-4}$ continuous?

ex. Evaluate based on continuity of the function:

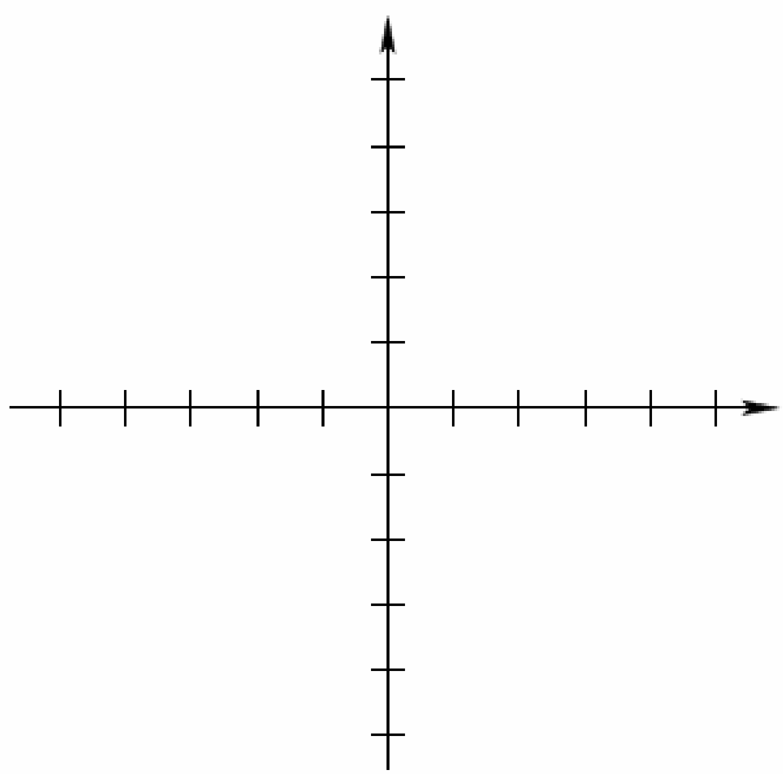
$$\lim_{x \to 1} 2^{x - 2\sqrt{x}}$$

So now we can use direct substitution to find:

ex. $\displaystyle \lim_{x \to \frac{\pi}{2}} \sin(\, x - \cos x \,)$

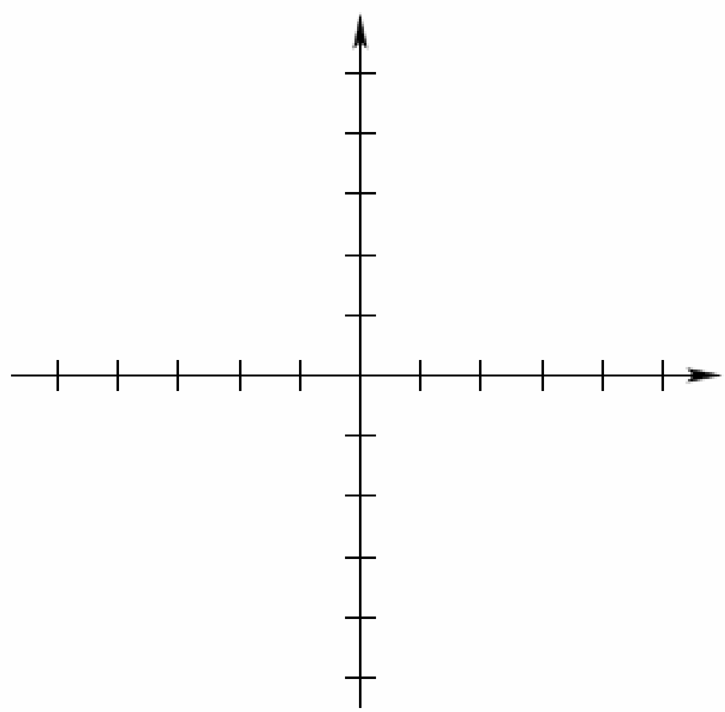
Functions with Discontinuities

ex. Consider $f(x) = \dfrac{x - x^2}{x^3 + x^2 - 2x}$. What types of discontinuities does $f(x)$ have? Could we define $f(x)$ to make it continuous at any of those discontinuities?



ex. Find any removable or nonremovable discontinuities of

$$f(x) = \begin{cases} x - 1 & \text{if } x < 0 \\ |x| + 1 & \text{if } 0 \le x < 1 \\ 2\sqrt{x} & \text{if } x > 1 \end{cases}$$



<u>**Def.**</u> A function f is **continuous from the right** at $x = c$ if

f is **continuous from the left** at c if

In our example, is f continuous from the right or left at $x = 0$?

At $x = 1$?

Continuity on Intervals

A function is continuous on an interval I if it is continuous at each x-value in I. If f is defined only on one side of an endpoint of the interval, we consider continuity to be continuity from the left or right.

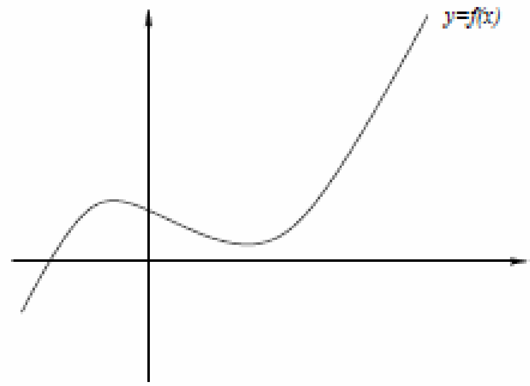
On an interval for which a function is continuous its graph has no jumps, holes, or gaps.

ex. Find the value of K which will make $f(x)$ continuous at $x = e$.

$$f(x) = \begin{cases} \ln(2x - e) & x \leq e \\ 4x - K & x > e \end{cases}$$

An important property of continuous functions is the **Intermediate Value Theorem (IVT)**:

Suppose f is continuous on $[a, b]$ and M is any number between $f(a)$ and $f(b)$, then



Corollary. (Existence of Zeros)

If $f(x)$ is continuous on $[a, b]$ and if $f(a) > 0$ and $f(b) < 0$ (or $f(a) < 0$ and $f(b) > 0$), then $f(x)$ has a zero in (a, b).

<u>**ex.**</u> Does the IVT imply that $f(x) = \dfrac{1}{x}$ has a zero on $(-1, 1)$?

<u>**ex.**</u> Use the IVT to show that the equation $x^3 + 3x = 2$ has a root in the interval $(-1, 1)$.