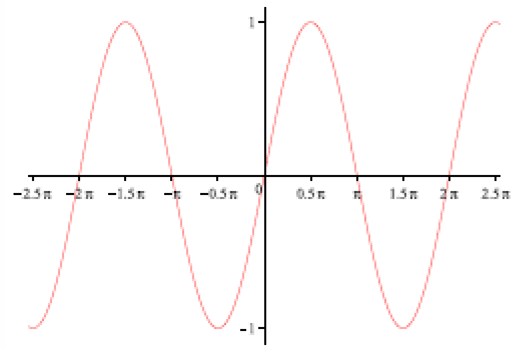
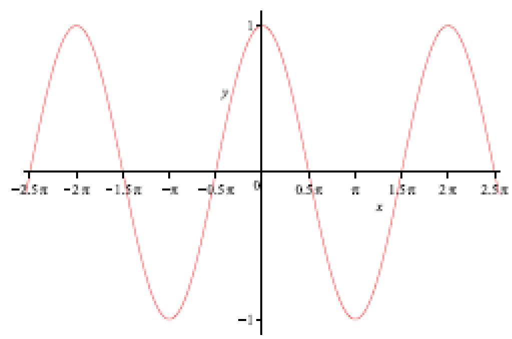
L4 Trig Limits

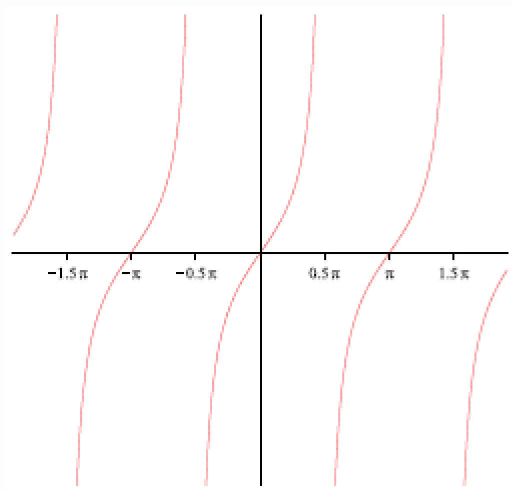
1) $\lim_{\theta \to 0} \sin \theta =$



2) $\lim_{\theta \to 0} \cos \theta =$



3) $\lim_{\theta \to 0} \tan \theta =$



ex. $$\lim_{x\to 0} \frac{\cos x - 1}{\sec x - 1}$$

ex. $$\lim_{x\to 0^-} \frac{\sin x}{1 - \cos x}$$

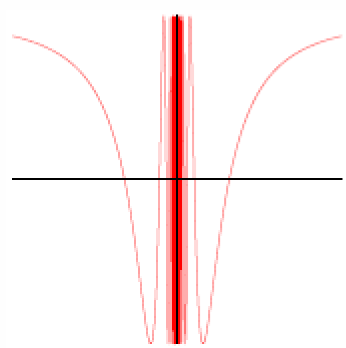
Theorem. (Squeeze Theorem)

If f, g and h are functions so that $f(x) \le g(x) \le h(x)$ for all x near c(except possibly at c) and if
$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$,

then

ex. Find $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$.

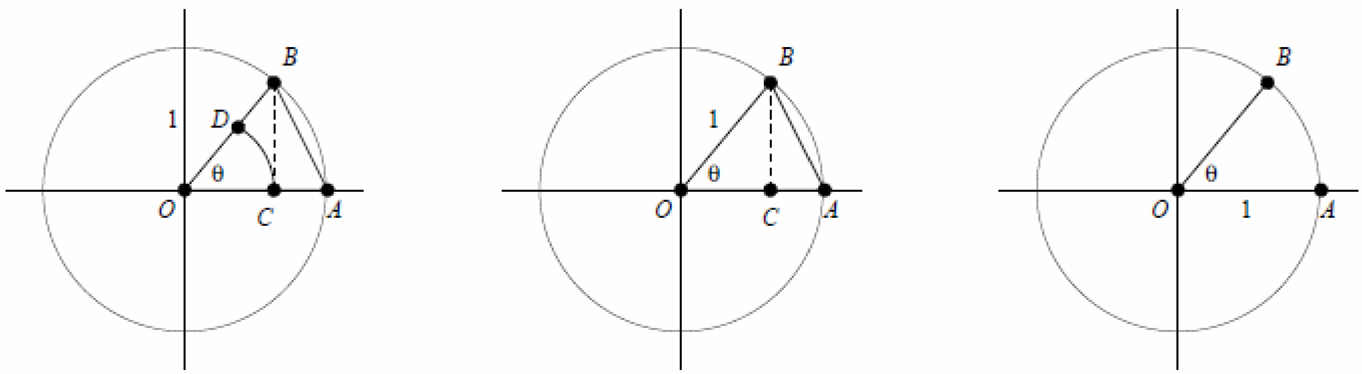
1) First consider $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$.



2) Evaluate $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$.

Recall: Area of a sector with a central angle of θ and radius of r is

Theorem. $\displaystyle \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$



area of sector COD $<$ area of $\triangle BOC$ $<$ area of sector AOB

Corollary. $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$

Corollary. $\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$

Now we can use these limits to find the following:

ex. $$\lim_{x \to 0} \frac{\sin(8x)}{x}$$

ex. $$\lim_{x \to 0} \frac{\sin^2(\pi x)}{3x^2}$$

ex. $\lim_{x \to 0} \frac{\sin^3(2x)}{\tan^3(4x)}$

ex. $\lim_{x \to 0} \frac{1 - \cos(2x)}{x^2}$

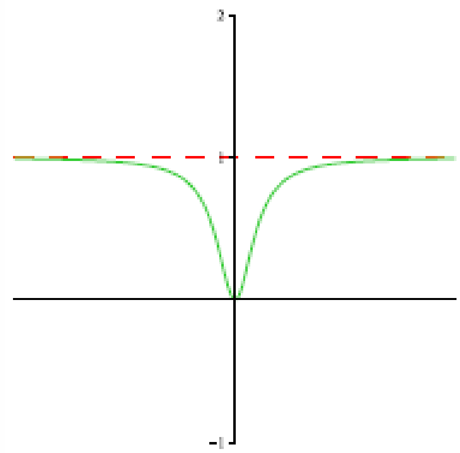
Evaluate the following limits using substitution:

ex. $$\lim_{x \to \frac{\pi}{2}} \frac{\sin(\cos x)}{\cos x}$$

ex. $$\lim_{x \to \pi} \frac{\sin x}{x - \pi}$$

Limits at infinity

Consider the graph of $f(x) = \dfrac{x^2}{x^2+1}$:



What happens to $f(x)$ as x increases in absolute value?

$$\lim_{x\to\infty} f(x) = \qquad\qquad \lim_{x\to-\infty} f(x) =$$

Def. $\lim_{x\to\infty} f(x) = L$ if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

We have similar definitions as $x \to -\infty$.

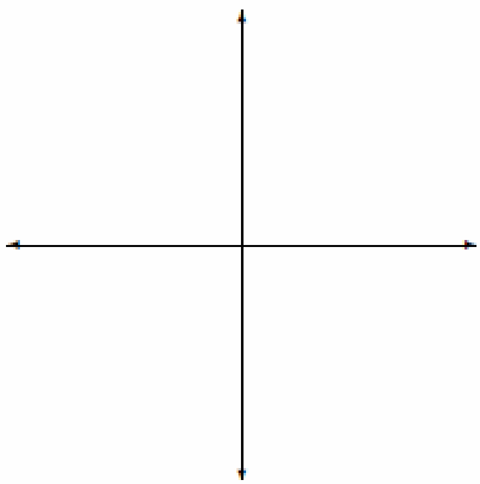
Def. $\lim_{x\to-\infty} f(x) = L$ if the values of $f(x)$ can be made arbitrarily close to L by taking x to be negative but sufficiently large in absolute value.

Def. The line $y = L$ is called a **horizontal asymptote** of the graph of $f(x)$ if

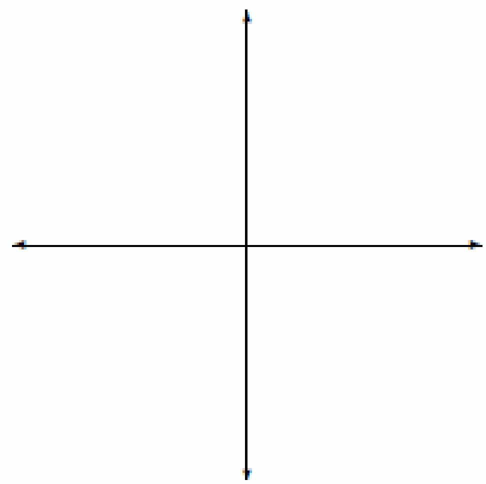
How many horizontal asymptotes can a graph have?

Consider the following functions:

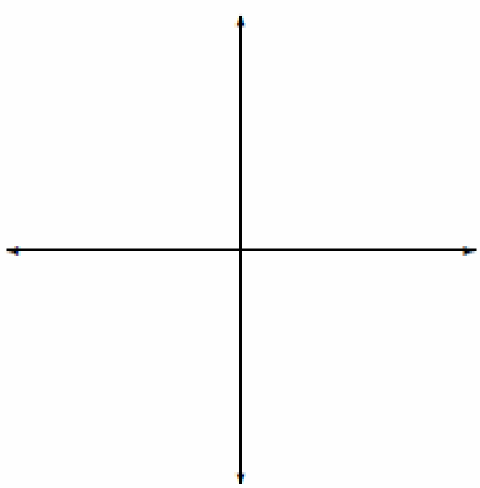
ex. $f(x) = x$



ex. $f(x) = 1 - e^x$



ex. $f(x) = \tan^{-1}(x)$



$$\lim_{x\to\infty} \tan^{-1}(x) =$$

$$\lim_{x\to-\infty} \tan^{-1}(x) =$$

Infinite Limits at Infinity

Def. $\lim_{x\to\infty} f(x) = \infty$ if we can make $f(x)$ as large as we want by choosing x sufficiently large.

We have similar definitions for $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

To evaluate limits at infinity

The limit laws still apply, and are often used with the following theorem to evaluate limits at infinity.

Theorem. If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} =$$

and

$$\lim_{x\to-\infty} \frac{1}{x^r} =$$

provided $\frac{1}{x^r}$ is defined when $x \to \infty$ or $x \to -\infty$.

ex. $\lim_{x\to-\infty} \frac{3x(2x^2+1)}{6-2x^3} =$

Does the graph of $f(x) = \frac{3x(2x^2+1)}{6-2x^3}$ cross its horizontal asymptote?

Shortcut for finding limits at infinity for rational functions

Theorem.

If $f(x) = \frac{p(x)}{q(x)}$

where $p(x)$ is of degree n and $q(x)$ is of degree m, then

1) If $m > n$, $\lim_{x\to\infty} f(x) =$

2) If $n = m$, $\lim_{x\to\infty} f(x) =$

3) If $m < n$, $\lim_{x\to\infty} f(x) =$ and $\lim_{x\to-\infty} f(x) =$

ex. Find all asymptotes of $f(x) = \dfrac{4 - x^2}{x^2 + x - 2}$.

Does the graph cross its horizontal asymptote?

ex. Find any horizontal asymptotes of $f(x) = \dfrac{3x}{\sqrt{4x^2 - 1}}$.

What is the domain of $f(x)$?

ex. $\lim_{x\to\infty} \frac{3}{\ln(x-3)+4}$

ex. $\lim_{x\to\infty} \frac{\sin(x)}{x}$

ex. $\lim_{x\to\infty} (x^2 - x^3)$

Consider the following limits involving exponentials:

$$\lim_{x\to\infty} e^x = \qquad\qquad \lim_{x\to\infty} e^{-x} =$$

$$\lim_{x\to-\infty} e^{\frac{1}{x}} =$$

ex. Find all asymptotes of $f(x) = \dfrac{2e^x}{3e^x - 4}$.

We can use the shortcut for finding limits at infinity for rational functions on the test:

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + \cdots}{b_m x^m + \cdots}$ be a rational function.

1) If $m > n$, $\lim_{x\to\infty} f(x) = 0$.

2) If $n = m$, $\lim_{x\to\infty} f(x) = \dfrac{a_n}{b_m}$.

3) If $m < n$, $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to-\infty} f(x) = \pm\infty$.

NOTE: We ignore the lower terms in both numerator and denominator in this shortcut method.

<u>ex.</u> Evaluate: $\lim_{x\to\infty} \dfrac{-x^4 + 3x}{x^2 + 1}$

$$\lim_{x\to\infty} \frac{-x^4 + 3x}{x^2 + 1} = \lim_{x\to\infty} \frac{\frac{-x^4}{x^4} + \frac{3x}{x^4}}{\frac{x^2}{x^4} + \frac{1}{x^4}} = \lim_{x\to\infty} \frac{-1 + \frac{3}{x^3}}{\frac{1}{x^2} + \frac{1}{x^4}} = \frac{-1}{0^+}$$

$$= -\infty \quad \text{(formal way to find the limit)}$$

$$\lim_{x\to\infty} \frac{-x^4 + 3x}{x^2 + 1} = \lim_{x\to\infty} \frac{-x^4}{x^2} = \lim_{x\to\infty} -x^2 = -\infty$$

(shortcut)

We can modify our shortcut method for an indeterminate limit $\dfrac{\infty}{\infty}$:

<u>**ex.**</u> Evaluate: $\displaystyle\lim_{x\to-\infty} \frac{x}{2x-\sqrt{4x^2-x}}$

$$(\text{formal method}) = \lim_{x\to-\infty} \frac{x}{2x-\sqrt{4x^2(1-\frac{x}{4x^2})}}$$

$$= \lim_{x\to-\infty} \frac{x}{2x-|2x|\sqrt{1-\frac{1}{4x}}}$$

$$= \lim_{x\to-\infty} \frac{x}{2x+2x\sqrt{1-\frac{1}{4x}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{x}{x}}{\frac{2x}{x}+\frac{2x}{x}\sqrt{1-\frac{1}{4x}}}$$

$$= \lim_{x\to-\infty} \frac{1}{2+2\sqrt{1-\frac{1}{4x}}}$$

$$= \frac{1}{4}$$

$$
\begin{aligned}
\text{(shortcut)} &= \lim_{x\to-\infty} \frac{x}{2x-\sqrt{4x^2}} \quad \text{Ignore the lower term(s)!} \\
&= \lim_{x\to-\infty} \frac{x}{2x-|2x|} \\
&= \lim_{x\to-\infty} \frac{x}{2x+2x} \\
&= \frac{1}{4}
\end{aligned}
$$

NOTE: We **cannot** ignore the lower terms for an indeterminate limit $\infty - \infty$:

<u>**ex.**</u> Evaluate: $\lim_{x\to\infty} 2x - \sqrt{4x^2+1}$

<u>**ex.**</u> Evaluate: $\lim_{x\to\infty} 2x - \sqrt{4x^2+2x+1}$