L5 Definition of the derivative

Tangent Lines and Slope

**Def.** The slope of the secant line through the point \( P(a, f(a)) \) and a nearby point \( Q(x, f(x)) \):

\[
m = \frac{f(x) - f(a)}{x - a}
\]

provided that the limit exists.

**Ex.** Find the equation of the tangent line to \( f(x) = x^3 - 1 \) at \( x = 2 \).
Alternate Definition of the Slope of a Tangent Line

Let \( h = x - a \). Then

and \( m = \)

**ex.** Find the slope of the tangent line to

\[
 f(x) = \frac{x}{x + 1} \at \text{at } x = 2. 
\]
NOTE: We can use our formulas to find the slope of the tangent line for $f(x)$ at any point $(a, f(a))$. 

ex. Find the slope of the tangent line to $f(x) = \sqrt{9 - 2x}$ at $x = a$. 
We can use this formula to find the slope of the tangent line to \( f(x) = \sqrt{9 - 2x} \) at a given value \( x = a \) (if the limit exists):

1) \( x = -8 \)

2) \( x = 0 \)

3) \( x = \frac{9}{2} \)
The derivative as a function

**Def.** Given \( y = f(x) \),

\[
f'(x) =
\]

The derivative is itself a function of \( f \). Its domain:

Other notations for the derivative:

Process of finding the derivative is called
ex. Find the function $f'(x)$ for $f(x) = \frac{1}{x - 2}$. What is its domain?

**Def.** A function $f$ is **differentiable** at $x = a$ if $f'(a)$ exists. It is differentiable on an open interval if it is differentiable at each number in the interval.

ex. Find each interval for which $f(x) = \frac{1}{x - 2}$ is differentiable.
ex. Find each $x$-value at which the tangent line to the graph of $f(x) = \frac{1}{x - 2}$ is perpendicular to the line $y = 4x$. 
Recall: the derivative of a function $f$ at $x = a$ is

$$f'(a) =$$

The derivative as a function

**Def.** Given $y = f(x)$, $f'(x) =

**ex.** Let $f(x) = \begin{cases} 1 - x & x < 1 \\ x^2 - 1 & x \geq 1 \end{cases}$.

1) Find $f'(1)$ if possible.

2) Find a formula for $f'(x)$, where $f(x) = \begin{cases} 1 - x & x < 1 \\ x^2 - 1 & x \geq 1 \end{cases}$.
3) Sketch the graph of $f(x) = \begin{cases} 1 - x & x < 1 \\ x^2 - 1 & x \geq 1 \end{cases}$.

What do you note about continuity and differentiability of $f(x)$ at $x = 1$?

**Theorem.** If $f$ is differentiable at $x = a$, then $f$ is continuous at $x = a$. 
ex. Find $f'(1)$ if $f(x) = (x - 1)^{1/3}$.

Def. The graph of a function $f(x)$ has a \textbf{vertical tangent line} at $x = a$ if

What about a \textbf{horizontal tangent line}?
When is a function not differentiable at a point?
We know that at \( x = a \), \( f'(a) \) gives the slope of the tangent line to the graph of \( f \) at the point \((a, f(a))\).

**ex.** Given the graph of \( f(x) \), sketch a possible graph of its derivative.
ex. Find all points $P$ on the parabola $y = x^2$ such that the tangent line at $P$ passes through the point $(0, -4)$. 