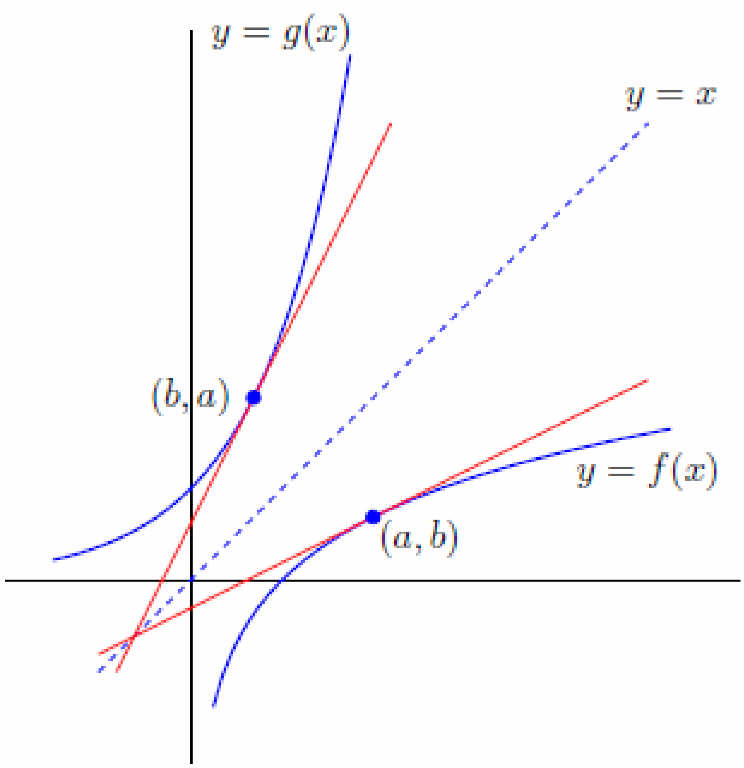
# L9 Derivatives of Inverse Functions

## Theorem. (Derivative of the Inverse)

Assume that $f$ is differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$. If $b$ is in the domain of $g$ and $f'(g(b)) \neq 0$, then $g'(b)$ exists and

$$g'(b) =$$



**ex.** If $f(x) = x^3 - 2$ and $g$ is the inverse of $f$, find

a) $g(x)$ and then find $g'(x)$ directly.

b) $g'(x)$ using the previous theorem.

Sometimes, we cannot find the inverse function explicitly.

<u>**ex.**</u> If $g$ is the inverse of $f(x) = 2x - \sin x$, find

a) $f(\pi)$

b) $g(2\pi)$

c) $g'(2\pi)$

<u>**ex.**</u> If $g$ is the inverse of a differentiable function $f(x)$ such that $f(-1) = 8$ and $f'(-1) = 12$, find $g'(8)$.

# Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) =$$

$$\frac{d}{dx}(\cos^{-1} x) =$$

$$\frac{d}{dx}(\tan^{-1} x) =$$

$$\frac{d}{dx}(\cot^{-1} x) =$$

$$\frac{d}{dx}(\sec^{-1} x) =$$

$$\frac{d}{dx}(\csc^{-1} x) =$$

**ex.** Let $f(x) = \sin^{-1}(2x - 1)$. Find the following:

a) $f'(x)$

b) domain of $f'(x)$

c) domain of $f(x)$

# Derivatives of General Exponential Functions

Recall $\dfrac{d}{dx}(e^x) =$

$\dfrac{d}{dx}(b^x) =$ for $b > 0$

Write $f(x) = b^x =$

**ex.** Find the slope of the tangent line to $f(x) = 2^{\sec^{-1}(x)}$ at $x = \sqrt{2}$.

## Derivatives of Logarithmic Functions

$$\frac{d}{dx}[\log_b(x)] =$$

$$\frac{d}{dx}[\ln(x)] =$$

By the Chain Rule, if $g$ is differentiable,

$$\frac{d}{dx}[\ln(g(x))] =$$

or if $u$ is a differentiable function of $x$,

$$\frac{d}{dx}(\ln(u)) =$$

**ex.** Find $\dfrac{d}{dx}[\ln(x^3)]$

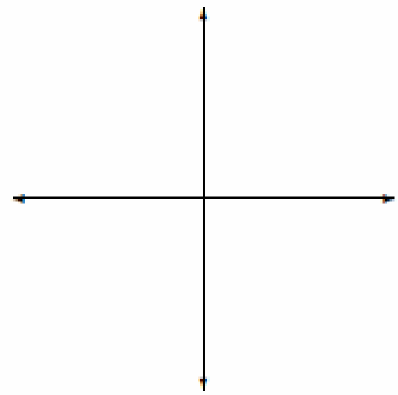
**ex.** Find $\dfrac{d}{dx}[(\ln x)^3]$

**ex.** Find $f'(x)$ if $f(x) = \ln(\tan^{-1}(e^x))$.

Be careful of domain!

**ex.** Find the horizontal tangent lines of $f(x) = \ln(x^3 - 3x)$.

**ex.** Find $f'(x)$ if $f(x) = \ln|x|$.



**ex.** Find the horizontal tangent lines of $f(x) = \ln|x^3 - 3x|$.

# Log and implicit differentiation

## Explicit Functions

<u>**ex.**</u> Find $\dfrac{dy}{dx}$ if $y - 2x = e^{\cos x}$.

## Implicit Functions

<u>**ex.**</u> Consider the equation $x^2y - 4x = y^3$.
If $y$ is a differentiable function of $x$, can we find $\dfrac{dy}{dx}$?

**NOTE:** This equation implicitly defines more than one function $y = f(x)$. We seek a formula for $\dfrac{dy}{dx}$ for all functions $f(x)$ satisfying the above equation.

Implicit Differentiation requires the Chain Rule.

Consider the following examples:

$$\frac{d}{dx}(x) =$$

$$\frac{d}{dx}(x^2) =$$

Now suppose that $y$ is a differentiable function of $x$.

$$\frac{d}{dy}(y^2) =$$

What is $\frac{d}{dx}(y^2)$?

## To Differentiate Implicitly:

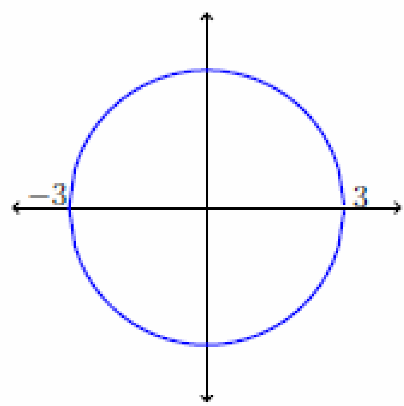
Assume $y$ is a differentiable function of $x$.

1. Differentiate both sides of the equation with respect to $x$.
2. Collect all terms involving $\frac{dy}{dx}$ on one side.
3. Rewrite by factoring out $\frac{dy}{dx}$.
4. Solve for $\frac{dy}{dx}$.

**ex.** Now we can find $\frac{dy}{dx}$ if $x^2y - 4x = y^3$.

**ex.** Find the slope of the tangent line to $x^2 + y^2 = 9$ at the point $(2, -\sqrt{5})$

a) Explicitly



Find the slope of the tangent line to $x^2 + y^2 = 9$ at the point $(2, -\sqrt{5})$

b) Implicitly

c) Find an expression for $\frac{d^2y}{dx^2}$.

**ex.** Find $\frac{dy}{dx}$ if $y = \tan(xy)$.

We can also use "Implicit Differentiation" to find the derivative of inverse functions.

For example, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$.

Recall the following: If $x > 0$ and $y > 0$,

1. $\ln(xy) =$
2. $\ln\left(\dfrac{x}{y}\right) =$
3. $\ln(x^y) =$

We can use these properties to write a complicated logarithmic function into a form involving sums and differences, which are easier to differentiate.

<u>**ex.**</u> Find $f'(x)$ if $f(x) = \ln\sqrt{\dfrac{x^2+2x}{2x-6}}$.

## Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.

2. Differentiate implicitly with respect to $x$.

3. Solve for $\frac{dy}{dx}$.

We can use the process of **Logarithmic Differentiation** to find the derivative of a complicated expression which does not contain logarithms initially:

**ex.** Find $f'(x)$ if $f(x) = \frac{\sqrt{2x-3}}{e^{2x}\sec x}$.

**NOTE:** For $a$, $b$ constants

1. $\dfrac{d}{dx}(a^b) =$

2. $\dfrac{d}{dx}([f(x)]^b) =$

3. $\dfrac{d}{dx}(a^{g(x)}) =$

4. $\dfrac{d}{dx}([f(x)]^{g(x)}) =$

**ex.** Find the equation of the tangent line to $f(x) = x^{\ln x}$ at $x = e$.