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On Classical Systems and Measurements in Quantum Mechanics

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Abstract The recent rigorous derivation of the Born rule from the dynamical law of quantum mechanics [Phys. Rep. 525(2013) 1-166] is taken as incentive to reexamine whether quantum mechanics has to be an inherently probabilistic theory.

It is shown, as an existence proof, that an alternative perspective on quantum mechanics is possible where the fundamental ontological element, the ket, is not probabilistic in nature and in which the Born rule can also be derived from the dynamics.

The probabilistic phenomenology of quantum mechanics follows from a new definition of statistical state in the form of a probability measure on the Hilbert space of kets that is a replacement for the von Neumann statistical operator to address the lack of uniqueness in recovering the pure states included in mixed states, as was pointed out by Schrödinger.

From the statistical state of a quantum system, classical variables are defined as collective variables with negligible dispersion. In this framework, classical variables can be chosen to define a derived classical system that obeys, by Ehrenfest's theorem, the laws of classical mechanics and that describes the macroscopic behavior of the quantum system.

The Born rule is derived from the dynamics of the statistical state of the quantum system composed of the observed system interacting with the measurement system and the role of the derived classical system in the process is exhibited. The approach suggests to formulate physical systems in second quantization in terms of local quantum fields to ensure conceptually equivalent treatment of space and time.

A real double-slit experiment, as opposed to a thought experiment, is studied in detail to illustrate the measurement process.

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1 Introduction

The formulation of quantum mechanics (QM) is considered to have been completed around 1930. However, the way QM describes classical physics and includes classical mechanics (CM) and the way experimental observations on quantum systems fit into the theory has not yet been resolved in a totally satisfactory way. There is general agreement on how to use QM correctly. The accomplishments, in the form of technologies that are part of modern life, are evidence that the approach is correct and effective. In contrast, there are legitimate and deep mathematical and philosophical questions that do not have satisfactory answers, or even a generally accepted approach to finding answers for them [50, 29, 91, 96]. The book by Greenstein and Zajonc [44] gives a beautiful summary of these issues including a detailed discussion of real experiments, as opposed to thought experiments, to provide context and evidence for this debate.

The problem of experimental measurements on microscopic quantum systems is inextricably linked to the problem of describing macroscopic classical systems, that satisfy classical laws, as an integral and logically consistent part of QM. The recent work by Allahverdyan, Balian, and Nieuwenhuizen [3] provides an excellent overview of the issues. It also formulates a mathematically detailed and logically precise solution to the problem. We will refer to the authors as "ABN" and use their work as the starting point for our discussion; we refer the reader to their work for a thorough review of previous work and alternative formulations.

Every theory in physics can be approached from two opposite ends: One can start from experiment and observations to formulate laws and procedures to predict the outcome of experiments. This is called the **empiricist view**, presented, for example, in the book by de Muynck [60]. Or, one can start from a minimal set of concepts and principles to formulate laws of Nature and rigorously derive the phenomenology that can be verified by experiments, as is done for classical physics in all textbooks, for example, on Newtonian mechanics. By the authority of experimental verification, the concepts are then viewed as the reality behind the phenomena, so that we may refer to this approach as the **realist view**. The knowledge of physics grows down from the empiricist end towards the foundations by increasing systematization and clarification, and it grows up from the realist end towards the empirical phenomena by precise, mathematical derivation from an assumed set of foundation principles. It can be said that the problem in the foundations of QM is the unresolved tension between the empiricist and realist views.

Thus far no consensus has been reached as to what set of assumptions allow for a rigorous and fully understood derivation and explanation of all quantum phenomena. All textbook expositions[1] of QM include the Schrödinger equation (SE), Eq, (A.11), and the mathematical framework of Hilbert spaces to formulate it. But there is a varying number of additional principles or assumptions to get to the theory for the working physicist [33, 61, 91], with the primary assumption to be added being Born's probability rule Eq. (A.14), in one form or another, to describe the outcome of the measurement process carried out on a quantum system.

This paper presents a derivation of the complete quantum phenomenology from the realist end using only the SE, with the Hilbert space context in which it is formulated, as the foundation. We consider the object controlled by the SE, namely the ket, to be as real as the "objects with substance" governed by Newton's Law of motion in classical mechanics.[2] We present a derivation that shows that the standard textbook presentation of QM [91], i.e. the formalism with the framework of associated rules, is correct. Thus, theorists can indeed claim that QM provides, contrary to d'Espagnat's caution [28, p. 404], an ontological foundation for QM, i.e. that the "completeness hypothesis" [28, §4.2] is correct.

ABN [3, 4] provide a detailed and rigorous derivation[3] from the Liouville von Neumann equation (LvNE), Eq. (B.10), of the process that is captured by Born's rule as a shortcut in a phenomenological way. Their description is minimalist, in that it does not involve special assumptions, and it approaches the measurement process from the empiricist point of view. It establishes, for the first time with full rigor, that the SE can and does describe all dynamics in QM. In other words, ABN prove that the formulation of the uncertainty principle [48] and the probability rule [18, 19] provide only a

[1] See Appendix A for a short formulation of QM.
[2] See Appendix C.1 for details.
[3] For ease of reference, the description of the measurement process derived by ABN is summarized in Appendix C.2.

phenomenological shortcut between theory and experiment, not a principle of foundation.

I interpret the result by ABN to mean that the digging from the empiricist end has hit a firm foundation for QM. That in turn warrants a fresh investigation from the realist end as to whether all phenomenology can be derived from a set of foundation principles. This inspires a constructive formulation for QM

1. that starts with a deterministic mechanics of kets, largely hidden from experiment (§2.1),
2. then puts a probability measure on top of the phase space of kets to get the probabilistic features of QM (§2.2), and
3. finally extracts a description of emergent, classical systems from the probability measure that is the statistical state (§2.3).

We will argue (§2.2) that the von Neumann density operator[4] is not quite right as a statistical probability distribution for kets as the basic dynamical object in QM. As pointed out early on by Schrödinger [75, 76,63,23], there is no unique set of kets and associated probabilities that can be recovered from a given density operator. Therefore, we will use a true probability measure on Hilbert space to describe the statistical state in QM instead.

To avoid confusion between classical systems described by classical theories of physics and the emergent systems in macroscopic quantum systems that satisfy the laws of classical physics, we will refer to the latter as **q-classical**, following the terminology of q-bit and qubit in quantum computing and q-probability introduced by ABN [4].

After presenting the construction of q-classical systems and their dynamical law within QM, we apply the formalism to an experimental realization of a double-slit experiment [86] as an example in Section 3. First we describe the experiment from the empiricist view, applying the work of ABN [3,4]. We analyze in Section 3.2 the role of the position observable for the particles in the experiment. We will argue that position is inadmissible as an observable, and that all physical systems need to be described in terms of fields so that position is a coordinate, like time, even in the case of non-relativistic QM.

The main result of the paper is then the realist explanation of what happens, in full detail, in the double slit experiment and, as will be obvious, by natural extension to any measurement on quantum systems:

1. A fully deterministic evolution of kets flows in Hilbert space.
2. The high-precision statistical state of the quantum system and the large-system statistical state of the measurement apparatus, consisting of many mini detectors, evolve together. The statistical state gives the probability of which mini detector is triggered to emit a signal strong enough to be amplified and recorded.
3. The macroscopic classical variables of the combined system, mostly the macroscopic apparatus, as collective variables determined by the statistical state evolve deterministically according the classical dynamical laws to produce the visible, recorded results of the measurement.

For the double-slit experiment, the electron quantum field goes through both slits and the detector randomly picks the location where the quantum field produces a localized recording of an electron excitation. A derivation of Born's rule follows in Sect. 3.4. The paper concludes, in Section 4, with a summary and outlook.

2 Realist formulation of statistical QM

Experiments on quantum systems are probabilistic so that QM is often said to be intrinsically probabilistic [50,27,42,91,3]. As a result, a full discussion of QM must take place in the context of statistical QM [7,60]. In this paper, QM is formulated with the same mathematical structure as statistical classical mechanics,[5] i.e. with a deterministic mechanics at the foundation and a probability measure on top of the phase space to describe the statistics. We motivate this approach as follows.

What if Einstein and Bohr were arguing about the wrong issue? Both were thinking in terms of an essentially classical ontology, which we call the **Newton-Maxwell ontology**.[6] This ontology had successfully absorbed the conceptual shifts introduced by relativity as well as some shifts forced by QM, as shown by their early success with photons [35] and the hydrogen atom [14], respectively. This seemed to indicate that the ontology could handle the "quantum conditions" as well. Then Heisenberg, Born, Jordan [47,21], Schrödinger [74,72,73], and Dirac [32] found the correct mathematical formulation of QM consisting of the specification of states and the dynamical law that governs their evolution. Heisenberg and Born, in their initial approach advocated to let the mathematics be the guide to solve the quantum problem [50,42]. In other words, they advocated for a new ontology, that we call the

[4] See Eq. (B.8) in Appendix B.3 for the definition.

[5] A brief summary of non-equilibrium statistical mechanics, both classical and quantum can be found in Appendix B.

[6] The definition is given in Appendix C.1.

Heisenberg-Dirac ontology.[7] We will follow their lead in this paper.

The initial concerns focused on the violation the Bell inequalities [10,60] implying nonlocality in QM in the sense that things can happen faster than the speed of light. Recent discussion in the literature around the loophole in Bell's theorem clearly established that such conclusions were unwarranted [62,55,56,30]. A second motivation for the approach taken in this paper comes from interpreting Bell's theorem, and its generalizations including the recently published theorem of Pusey, Barrett, and Rudolph [65], as the proof that the inner workings of QM do not fit a Newton-Maxwell ontology. In other words, we take it as encouragement to consider the ket as the fundamental and irreducible dynamical object for QM, as in the Heisenberg-Dirac ontology.

2.1 Deterministic dynamics of kets

Our formulation of QM starts, as in all textbooks,[8] by defining the Hilbert space of kets and adopting

1. Assumption A.1, kets are basic, and
2. Assumption A.2, the SE Eq. (A.11) governs their dynamics.

The SE propagates any ket in a fully deterministic way to time t from its initial condition at time s as given by the evolution operator Eq. (A.12)

$$\Psi_t = \boldsymbol{U}(t-s)\Psi_s. \tag{2.1}$$

We will take it as a physical reality that all kets evolve and keep evolving. They can become distorted by interactions with other kets; they will get entangled when composite systems are considered.[9] Kets do not collapse or disappear.

Observables are introduced as Hermitian or self-adjoint operators. With the spectral representation [67] of self-adjoint operators, the kets can be given a realization as complex-valued square-integrable functions on the combined spectrum of a complete set of commuting observables.

Although the notion of expectation value of an observable $\boldsymbol{A}$ in a ket $|\Phi\rangle$ can be computed as

$$a = \frac{\langle\Phi|\boldsymbol{A}|\Phi\rangle}{\langle\Phi|\Phi\rangle}, \tag{2.2}$$

we do not assume any meaning related to statistics or probability theory. Following ABN [4], we refer to such mathematical quantities as **q-expectations**.

Let us consider the issue of what an experimental observation of a ket would mean. Einstein and Bohr assumed that the experimental data obtained was the meaningful and complete information about the QM state. What if it is not? The formalism tells us that the ket (or the wave function as its representation on the spectrum of some observables) is the full specification of the state; a single number in the spectrum of the observable provided by a measurement is not a complete specification.

Consider the measurement of spin. A spinor is determined by two complex numbers as a vector in a complex two-dimensional Hilbert space. Allowing for the normalization $|c_1|^2 + |c_2|^2 = 1$, that means a spinor is specified by three real numbers. They can be meaningfully chosen as three Euler angles specifying the spinor orientation in a frame in three dimensional space [90, p. 94]

$$\begin{pmatrix} e^{-i\frac{1}{2}(\varphi+\psi)}\cos\frac{1}{2}\vartheta \\ e^{i\frac{1}{2}(\varphi-\psi)}\sin\frac{1}{2}\vartheta \end{pmatrix}. \tag{2.3}$$

However, the way spin is measured, for example by a Stern-Gerlach apparatus, only two numbers are obtained: $|c_1|$ and $|c_2|$. Can we build an experimental apparatus that would find the axis defined by the spinor and then produce the three Euler angles?

The situation gets worse with observables that have continuous spectrum, e.g. position. Lundeen, Sutherland, Patel, Stewart, and Bamber [59] carried out a beautiful experiment measuring the wave function of a photon (real and imaginary part) by scanning through the range of the observable position. A desirable quantum experiment would produce, in a single interaction, the full complex wave function $\Psi(x)$ on the entire spectrum, which is a faithful representation of the quantum state of the system. Such an experiment would be the conceptual equivalent to the measurement of the position and velocity of a classical system like a planet, leading to the full determination of its state (for the purposes of celestial mechanics). This does not exist for QM. The fact that experiments on quantum systems produce a small number of values effectively means that most of the quantum world remains hidden from experiment. Only theory and computation have a chance to see deeper.

To execute our plan of deriving the phenomenology of QM from the assumptions in Appendix A.1, we must admit that the measurement process is more complex. It is fair to say that every experiment in QM is a scattering experiment that involves systems with more

[7] See Appendix C.3 for the definition.

[8] A brief summary of the formulation of QM is given in Appendix A.

[9] The composition of systems and the reverse, extracting subsystems from systems, are two operations that show the big difference between CM and QM, as discussed in Appendix A.2 and A.3.

microscopic parts than there are macroscopic parts in the Large Hadron Collider detectors ATLAS [6] and CMS [25], namely a number of atoms and molecules of the order of Avogadro's number 10^{23}. While Born's rule is very effective and accurate, it does not explain how quantum dynamics leads to observed outcomes in macroscopic systems that are themselves supposed to be described by QM [23]. We need more structure than the mechanics of kets. To deal with large numbers of kets, we turn to a statistical description.

2.2 Probabilities and statistics

When von Neumann [61] introduced[10] the statistical operator Eq. (B.8), he intended to generalize the classical probability distribution on phase space that assigns a probability to different states. From the analysis in Theorem A.2, von Neumann saw the possibility to introduce a statistical state for quantum systems that aligns well with the Born probability rule. It takes the form

$$\boldsymbol{D} = \sum_{N=0}^{\infty} \sum_{k_1,\ldots,k_N} p_{k_1,\ldots,k_N} |\Phi, k_1, \ldots, k_N\rangle\langle\Phi, k_1, \ldots, k_N|. \tag{2.4}$$

von Neumann intended these operators to give the probability $p_{k_1,\ldots,k_N}$ that the system is in the state described by the ket $|\Phi, k_1, \ldots, k_N\rangle$. It is only after the Born rule is established that these numbers, called q-probabilities [4], can be identified with probabilities that have the usual frequency interpretation in probability theory.

The operator provides a way to describe a state that is not completely known, a mixture or Gemenge [23, p. 21], by assigning multiple possible kets or wave functions to the system, each weighed by a positive number that can be interpreted as the probability for the system to be in the state represented by the corresponding ket or wave function. Because states of combined systems naturally decompose into statistical operators by Theorem A.2, the state of the system consisting of an observed quantum system and a measuring apparatus naturally leads to the consideration of states for the component systems with such a structure. When the proper observable operator and associated apparatus is chosen, the outcome is in agreement with Born's rule[11] [18,19]. Thus, the proposal by von Neumann to take the statistical operator as the appropriate definition of statistical state in QM seems like the correct way to proceed.

Schrödinger [75,76,63,23] pointed out that the statistical operator cannot be written in a unique way as a mixture of multiple pure states, i.e. projectors on one-dimensional spaces $\boldsymbol{P} = |\Psi\rangle\langle\Psi|$, if one allows them to be not orthogonal. As a result a general mixed state can be created from a set of kets and associated probabilities, but once created it is not possible to reconstruct these kets and probabilities in a unique way. Therefore the probabilities associated with a mixed state cannot be uniquely separated from the probabilities associated with pure states, which is why the statistical interpretation [7] of QM is used by ABN [3]. For the statistical operator in QM to share with CM the property of being a unique construction from states, one would need to impose the orthogonality restriction to mixtures, which is not desirable since states in QM that are different do not have to be orthogonal kets.

Is the von Neumann statistical state the only and best way to define a statistical state in QM? In statistical CM, the statistical state is defined[12] as a probability measure on the classical phase space. To define the statistical state for systems in QM, we make the same construction as in statistical CM, but now on the quantum phase space $\mathbb{H}$. Then every state is assigned a unique probability without the need for different kets to be orthogonal.

2.2.1 System coordinates

A difference between defining systems in CM and QM pertains to the way coordinates can be chosen to describe a system, which becomes particularly important with large and macroscopic systems. Because CM deals with degrees of freedom that are specified by pairs of coordinate and momentum and can be composed as described in Theorem A.1, there is great freedom in choosing coordinates to describe systems. This freedom is further clarified by the fact that the dynamical equations of CM are invariant under canonical transformation, which can be very general. This generality is often useful to simplify the problem of describing the dynamics of a system and in finding solutions to the equations. One consequence is that it is possible to describe a classical system in layers: One can start by defining collective coordinates to describe a system at a coarse level and then add coordinates that provide additional details at multiple layers, with each layer describing a finer scale. By choosing the layers to align with the expected or observed dynamics of the system, the coupling between degrees of freedom in different layers can be minimized. This often can result in methods to obtain solutions with high efficiency and/or high accuracy.

[10] See Theorem A.2 in Appendix A for the inspiration for the form of the statistical operator.

[11] Given by Eq. (A.14) in Appendix A.

[12] Definition B.1 in Appendix B.

CM is scale invariant. Its concepts and equations apply equally well to the solar system, earthquakes, ocean flows, and man-made mechanical devices. This is not the case with QM, as pointed out by Dirac [33, p. 4], where there is the concept of the "absolute small." At first sight, the fact that a theory has a notion of absolute scale does not have to be connected to the ability to choose degrees of freedom; it could still be possible that the theory can be used with equal ease and validity at multiple scales. However, that is not the case with QM.

Because the union and separation of systems in QM do not form a group of operations as in CM, as shown by Theorem A.2, there is no freedom to define degrees of freedom with arbitrary and general coordinates. The way a system is composed matters. It is not possible to describe the dynamics of collective degrees of freedom independently of the degrees of freedom that make up the collective coordinates, as in CM, except in a small number of exceptions.[13]

If the description of a physical system requires quantum accuracy, then the system should be described consistently at that level for the degrees of freedom to interact naturally: The atomic hypothesis needs to be invoked and atomic degrees of freedom must be used and combined to describe the system. For example, in QM one cannot describe a planet as a system with a single degree of freedom,—the center of mass coordinate and momentum,—and a large mass denoting the Newtonian substance of the planet. If quantum accuracy is required in the description of the planet's motion,—which admittedly is an unlikely requirement,—the planet must be described in terms of the degrees of freedom of the constituent atomic nuclei and electrons. Only then are the degrees of freedom correctly combined. With that choice of degrees of freedom, the quantum dynamics of the planet will not exhibit Schrödnger cat states: The interactions between the very large number of degrees of freedom prevent that.

In other words, because QM has an absolute scale and because of limitations on combining and separating systems,—by tensor product instead of direct sum,—the quantum description of systems must be built from the bottom up.

[13] One such exception the case of atomic nuclei in the realm of atomic, molecular, and materials physics: It is not necessary to treat the coordinates of the protons and neutrons, or quarks and gluons, inside the nucleus explicitly to obtain highly accurate results in atomic, molecular, and materials physics. The center of mass coordinate of the protons and neutrons decouples and can be used by itself to describe the nucleus in that realm.

2.2.2 Statistical state = probability measure

Another difference is that the phase space in QM is an infinite-dimensional Hilbert space $\mathbb{H}$, instead of the finite-dimensional space $\mathbb{H} = \mathbb{R}^{2N}$ in CM. Hence we require the mathematical notion of a finite measure on an infinite dimensional space[14] [80,11].

Because there is no equivalent to Lebesgue measure on infinite-dimensional spaces,[15] we cannot define a probability distribution function (PDF) relative to it, but we can define a probability measure directly.

Definition 2.1 The **statistical state** σ of a quantum system is a probability measure on the phase space $\mathbb{H}$, the space of kets, or wave functions, of the quantum system.

Now the concepts from Appendix B, including normalization Eq. (B.2) and dynamical flow Eq. (B.4) can be carried over immediately to the phase space $\mathbb{H}$ in QM with the ket $|\Psi\rangle$ or the wave function Ψ replacing the classical state (q, p). The SE drives the dynamics of wave functions Ψ_t and defines the **Schrödinger flow**

$$|\Psi, t\rangle = \boldsymbol{U}(\mathbf{0}, t, I)|\Psi, 0\rangle, \tag{2.5}$$

similar to the Hamilton flow in Eq. (B.3). The flow can be extended naturally to Borel sets S in $\mathbb{H}$ as $S_t = \boldsymbol{U}(\mathbf{0}, t, I)S$, where $\boldsymbol{U}(\mathbf{r}, t, \Lambda)$ is the Poincaré group representation defined as the straightforward generalization of the unitary representation of translation in time. That flow defines the time evolution of the statistical state σ_t

$$\sigma_t(S) = \sigma(S_{-t}) = \sigma\big(\boldsymbol{U}(\mathbf{0}, -t, I)S\big). \tag{2.6}$$

in complete analogy with in Eq. (B.4).

In the theory of Lebesgue measure, sets S and subspaces $\mathbb{R}^m$ of dimension m less than the dimension of the space $\mathbb{R}^N$ on which the measure is defined, all have measure zero [67,81]. For example, points have measure zero in classical phase space. Similarly, individual kets on QM have measure zero for a statistical state σ. In other words, eigenstates have probability zero with our definition of statistical state. This is not a problem, because any experiment will work at all times with sets of kets that do have a nonzero probability, smaller for smaller sets of kets that specify a more precisely prepared state for a quantum system.

This needs to be contrasted with the von Neumann density operator. Because of its provenance by Theorem A.2 from the analysis of states for composite systems,

[14] In Appendix D we summarize a few of the most relevant properties pertinent to the formulation of statistical QM.

[15] See Theorem D.1 in Appendix D.

it assigns finite probabilities to the basis kets of the Schmidt decomposition. It is this property that shows that the density operator is not a true probability measure on the space of kets, but on linear subspaces [51, 88].

Furthermore, when we look at the Hilbert space $\mathbb{H}$ of all states, it becomes clear that, with respect to any choice of basis for each of the single degree-of-freedom subspaces, the number of entangled states far outnumbers the special superpositions that are tensor product states. In any statistical state σ the set of tensor product kets and all their superpositions have dimension less than the total Hilbert space of all kets and therefore form a set of measure zero. This means that any quantum system in any statistical state σ has all degrees of freedom of all subsystems entangled all the time.[16]

2.2.3 High-precision statistical state

The path to make the connection between the formalism of QM, the kets evolving under the SE, with experimental observations goes through the domain of statistical states. The first type of statistical state we will need is one that specifies a single ket with high precision, i.e. with little variance. Let the ket be $|\Theta\rangle$. Equivalently, one can think of the wave function associated with the ket in the spectral representation of any observable operator, such as field strength $\boldsymbol{\varphi}_{\mathbf{r}}$ in a point $\mathbf{r} = (x, y, z)$ so that the wave function takes the form $\Theta(\varphi_{\mathbf{r}})$.

We define a Gaussian measure γ on $\mathbb{H}$ for this statistical state. The ket $|\Theta\rangle$ is chosen as the mean. To specify the covariance operator Γ, we need to characterize the allowed fluctuations of the ket. This is most conveniently done by specifying a basis of kets $(|\Xi, n\rangle)_n$ and a variance γ_n for each one, as the eigenvalue of Γ associated with the eigenvector $|\Xi, n\rangle$

$$\Gamma|\Xi, n\rangle = \gamma_n|\Xi, n\rangle. \tag{2.7}$$

To make Γ positive definite, all eigenvalues γ_n must be chosen real and positive; to ensure finite trace, we require

$$\mathrm{Tr}\Gamma = \sum_{n=0}^{\infty} \gamma_n < \infty. \tag{2.8}$$

Observe that the Cameron-Martin space[17] of the measure γ is the set of all kets for which

$$\langle\Psi|\Gamma^{-1}|\Psi\rangle = \sum_{n=0}^{\infty} \frac{|\langle\Xi, n|\Psi\rangle|^2}{\gamma_n} < \infty. \tag{2.9}$$

That means that if the eigenvalues γ_n go to zero slowly, while still retaining a finite sum e.g. like n^{-2}, there will be many more kets $|\Psi\rangle$ in the Cameron-Martin space than when the γ_n go to zero fast, e.g. like $(n!)^{-1}$. Since the Cameron-Martin space is a set of measure zero, any vector in it counts as being the same as the mean $|\Theta\rangle$; the kets that contribute to the measure and to any integrals taken with it lie outside the Cameron-Martin space. It follows that a tight covariance operator, i.e. with γ_n going to zero rapidly with n, makes for a tight statistical state with mean $|\Theta\rangle$ and small variance; a loose Γ makes for a fuzzy statistical state with large variance.

High-precision statistical states will be used to describe quantum systems in kets that are well-known and experimentally well-characterized.

How are these statistical states realized in experiments? Consider two examples of the photoelectric effect: an electron extracted from a glowing wire by a large voltage and an electron excited from a core orbital by absorption of an X-ray photon. In both cases the final energy and velocity of the photo electron can be similar by proper choice of voltage. In the Newton-Maxwell ontology, one is inclined to think of the electron as a point particle and one envisions the final wave function in the two cases to be very similar and equal to a plane wave with the appropriate momentum. Close examination of the processes, mindful of the Heisenberg-Dirac ontology, leads to a distinction between the two cases. In the case of the high-voltage extraction, the electron is likely to come from the valence band of the metal wire, which means it is described by a wave function extended over several unit cells of the metal. The process of the photoelectric effect in this case, as described by the SE, transforms this extended, traveling wave inside the conducting metal into a higher-momentum, extended, traveling wave in free space. In the case of the X-ray photo electron, the initial wave function is a tightly bound core orbital and the evolution described by the SE produces a much more localized wave function traveling in free space.

The question arises as to how similar these two final wave functions really are? In perturbation theory treatments of the process, one uses a plane wave basis and clearly obtains quite satisfactory results in both cases. The answer lies in that one does not have access to or control over individual wave functions, or kets; experimental control can only be exercised on statistical states. In carefully controlled experiments like the two examples of the photoelectric effect, one creates high-precision statistical states. The variance of the statistical state is described by the correlation operator Γ. It introduces a fuzziness around the wave functions cre-

[16] The definition of superposition and entanglement can be found in Appendix A.3.

[17] See Appendix D for the definition.

ated by the two distinct processes that may require extremely fine and precise experiments to overcome.

2.2.4 Large-system statistical state

Next we consider the construction of a statistical state for a very large system, such as a macroscopic measurement apparatus. The system does not have to be homogeneous, but it may consist of parts that are metal and parts that have very complex chemical composition such as the detector in the double-slit experiment to be discussed in Section 3, which consists of a fluorescent film attached to optical fibers. To describe the statistical state, we will again build a Gaussian measure, which means we have to specify the mean ket $|\Theta\rangle$ and the covariance operator Γ. We will specify it through its eigenvectors $|\Xi, n\rangle$ and eigenvalues γ_n as before.

We consider a large system to be one with a number of degrees of freedom of the order of Avogadro's number. There is no way to capture the state, ket, or wave function of such a system experimentally; at best we can construct a theoretical and computational characterization with the uncertainties captured in the statistical state. Even the statistical state cannot be pinned down precisely by any experiment. Fortunately, the nature of measures on infinite dimensional spaces is such that high precision specification of the measure is not relevant for physics.

To construct the mean ket $|\Theta\rangle$ of a statistical state for a large system, we can start with the ket for a meaningful building block of the large system, such as, for example, a ket for a molecule or for a nanostructure, and with these kets compose a product ket for the large system. The basis $|\Xi, n\rangle$ can be chosen the same way to specify the type of deviations from the mean we wish to consider in the statistical state. Because of the uncertainty around the precise ket for the system, the eigenvalues γ_n will be chosen to decay slowly with n, for example like $n^{-(1+\varepsilon)}$ with $\varepsilon > 0$ so that the trace $\sum_n \gamma_n < \infty$. Any superposition

$$|\Theta\rangle + \sum_{n=0}^{\infty} c_n |\Xi, n\rangle \tag{2.10}$$

such that

$$\sum_{n=0}^{\infty} \frac{|c_n|^2}{\gamma_n} < \infty \tag{2.11}$$

will be in the Cameron-Martin space of the statistical state and be indistinguishable, by measure, from $|\Theta\rangle$. With such a slow decay for γ_n that space will include a lot of superpositions. That means that long-range (but not infinite range) extended states that have correlated and entangled molecules or nanostructures are described by the statistical state as indistinguishable from the mean state. As a consequence, the statistical state of a macroscopic system is an extremely high-quality generator of random kets for the macroscopic system. In other words, the macroscopic system is not in a single ket at all. This, as we will see in Sect. 3, is precisely where the randomness in QM emerges in our approach.

2.3 Emergent, deterministic q-classical systems

Statistical states give a probability distribution on all kets of the quantum system. Macroscopic systems have a large number of degrees of freedom, in the order of Avogadro's number 10^{23}. It is impossible in practice to measure and characterize the ket or wave function of such a large system. For example, it may include a microscopic quantum system and a macroscopic measurement apparatus. We are not interested in the full details of this statistical state, just in a limited number of collective, macroscopic variables that allow control over the system to be measured and a few more that provide information from the interaction between the apparatus and the observed system.

2.3.1 Collective variables

We follow the same line of thought as in Appendix B.2 to define collective variables for the statistical quantum state σ. The considerations on the frequency interpretation for statistical states of classical systems apply equally to macroscopic quantum systems, so that it is plausible to take the same view and read Definition 2.1 to mean:

1. In non-equilibrium statistical QM, the mathematical description of the state of a macroscopic physical system with N degrees of freedom is the statistical state of the system, i.e. the probability measure σ on the system's Hilbert space $\mathbb{H}$.
2. The macroscopic observables (volume, pressure, temperature) and their dynamical evolution are encoded in the finite measure σ and its dynamics σ_t derived by Eq. (2.6) from the underlying flow

$$|\Psi_t\rangle = \boldsymbol{U}(\mathbf{0}, -t, I)|\Psi_0\rangle \tag{2.12}$$

in $\mathbb{H}$, Eq. (2.5).

The question arises as to how one can effectively specify and characterize the measure σ? Since it is a measure on $\mathbb{H}$, which contains all N degrees of freedom, with N of the order of Avogadro's number $N_A =$

6×10^{23}, the full experimental specification of the measure is out of reach; we will have to rely on theoretical considerations to find a reasonable characterization. The number of variables that are accessible to experiment (pressure, volume, temperature, charge) is tiny compared to the number of degrees of freedom, but their behavior is crucial and can be verified experimentally. Hence, we seek to specify the small set of macroscopic variables in terms of the large number of degrees of freedom.

2.3.2 Q-classical variables

Taking into account that the Born rule gives the correct phenomenological description, we are lead to make two definitions.

Definition 2.2 Let $\boldsymbol{A}$ be an observable operator, self-adjoint, for a quantum system with Hilbert space $\mathbb{H}$ and let $|\Psi\rangle$ be a ket representing the quantum system. The real numbers

$$v_{\boldsymbol{A}}(|\Psi\rangle) = \langle\Psi|\boldsymbol{A}|\Psi\rangle \tag{2.13}$$

and

$$w_{\boldsymbol{A}}(|\Psi\rangle) = \left(\langle\Psi|\boldsymbol{A}^2|\Psi\rangle - v_{\boldsymbol{A}}(|\Psi\rangle)^2\right)^{\frac{1}{2}} \tag{2.14}$$

are called the **q-expectation** and the **q-variance**, respectively, of the observable $\boldsymbol{A}$ in the ket $|\Psi\rangle$.

Because we do not view the state or ket to have any probabilistic meaning or interpretation by assumption A.1, the two numbers have only meaning as a computational prescription, as will become clear next. We follow the nomenclature introduced by ABN [4] to distinguish the mathematical objects defined here from the very similar-looking counterparts in probability theory and statistics.

Definition 2.3 Let $\boldsymbol{A}$ be an observable operator for a quantum system with Hilbert space $\mathbb{H}$ and σ a measure on $\mathbb{H}$ representing a statistical state of the quantum system. The real numbers

$$\begin{aligned} m_{\boldsymbol{A}}(\sigma) &= \int_{\mathbb{H}} v_{\boldsymbol{A}}(|\Psi\rangle)\sigma(\mathrm{d}|\Psi\rangle) \\ &= \int_{\mathbb{H}} \langle\Psi|\boldsymbol{A}|\Psi\rangle\sigma(\mathrm{d}|\Psi\rangle) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \Delta_{\boldsymbol{A}}(\sigma) &= \left(\int_{\mathbb{H}} w^2_{\boldsymbol{A}}(|\Psi\rangle)\sigma(\mathrm{d}|\Psi\rangle)\right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{H}} \left(\langle\Psi|\boldsymbol{A}^2|\Psi\rangle - \langle\Psi|\boldsymbol{A}|\Psi\rangle^2\right)\sigma(\mathrm{d}|\Psi\rangle)\right)^{\frac{1}{2}} \end{aligned} \tag{2.16}$$

are the **expectation value or mean** and the **variance**, respectively, of the observable $\boldsymbol{A}$ in the statistical state σ.

Here the mean and variance have the familiar probabilistic meaning in mathematics because σ is a probability measure on $\mathbb{H}$. However, the meaning in physics is still not clear, because of the appearance of the q-expectation and q-variance; we have not specified their operational meaning in any experiment; we only have a theoretical expression.

As a self-adjoint operator, $\boldsymbol{A}$ has a spectral transform $\mathscr{A}$ [66]

$$\mathscr{A} : \mathbb{H} \to L^2(\mathrm{spec}(\boldsymbol{A}), \mathbb{C}) \tag{2.17}$$

that brings $\boldsymbol{A}$ to the form of a multiplication operator and a spectral representation

$$\boldsymbol{A} = \int_{\mathrm{spec}(\boldsymbol{A})} a\boldsymbol{M}_{\boldsymbol{A}}(\mathrm{d}a) \tag{2.18}$$

with the projection-operator-valued measure on the spectrum $\mathrm{spec}(\boldsymbol{A})$ of $\boldsymbol{A}$

$$\boldsymbol{M}_{\boldsymbol{A}}(S) : \mathbb{H} \to \mathbb{H} : \Psi \mapsto \boldsymbol{M}_{\boldsymbol{A}}(S)\Psi = \mathscr{A}^{-1}\iota_S\mathscr{A}\Psi \tag{2.19}$$

for measurable sets S in the spectrum. Here ι_S is the index function of the set S, equal to 1 for points in S and 0 otherwise. Then the probability measure for the values a of the observable $\boldsymbol{A}$ for the system in the statistical state σ is given by

$$\sigma_{\boldsymbol{A}}(S) = \sigma\left(\boldsymbol{M}_{\boldsymbol{A}}(S)\mathbb{H}\right). \tag{2.20}$$

The mean $m_{\boldsymbol{A}}(\sigma)$ and variance $\Delta_{\boldsymbol{A}}(\sigma)$ in Definition 2.3 are the mean and variance of this probability measure.

The above definitions apply to any observable, but they are only really useful for a special subset of observable operators in QM, namely ones that depend on a very large number of degrees of freedom, such as collective variables. If an observable $\boldsymbol{A}$ is the sum of N observables $\boldsymbol{A}_n$, then the q-expectation $v_{\boldsymbol{A}}(|\Psi\rangle)$ is the average of the q-expectations and the q-variance is smaller than the q-variances $w_{\boldsymbol{A}_n}(|\Psi\rangle)$ by a factor $1/\sqrt{N}$ [3]. Therefore, while observable operators in QM are in general not dispersion free [51], observable operators that are collective variables can have very small variances if N is sufficiently large. Collective observables then behave very much like classical variables in that they take on values without dispersion [51]. This motivates the following definition.

Definition 2.4 Let $\boldsymbol{A}$ be an observable operator for a quantum system with Hilbert space $\mathbb{H}$ and σ a measure on $\mathbb{H}$ representing a statistical state of the quantum system. When the variance $\Delta_{\boldsymbol{A}}(\sigma)$ is sufficiently small, the mean $m_{\boldsymbol{A}}(\sigma)$ is called a **classical variable associated with the quantum system** or, shorter, **q-classical variable**.

In practice, if the variance is smaller than the experimental error, it can safely be considered sufficiently small and the mean of the observable operator is a q-classical variable.

2.3.3 Q-classical systems

In the description of macroscopic systems, we will typically identify a set of collective coordinates Q and associated velocities $\dot{Q}$ expressed in terms of a very large number of observable operators associated with the underlying microscopic degrees of freedom.

Definition 2.5 The system of degrees of freedom consisting of coordinates and associated velocities or conjugate momenta that all are q-classical variables is called a **classical mechanical system derived from the quantum system** or, shorter, a **q-classical system**.

The evolution of the statistical state is governed by Eq. (2.6) and follows the Schrödinger flow of QM. The mean in the statistical state of all q-classical variables then evolves as dictated by the Ehrenfest theorem [34].

Theorem 2.1 (Ehrenfest) *The q-classical variables in a q-classical system can be chosen to satisfy Hamilton's equations*

$$\begin{aligned} \dot{Q} &= \frac{\partial}{\partial P} H(P,Q) \\ \dot{P} &= -\frac{\partial}{\partial Q} H(P,Q). \end{aligned} \tag{2.21}$$

It follows that the q-classical systems consisting of the q-classical variables $(Q, \dot{Q})$ (Lagrange) or (Q, P) (Hamilton) satisfy classical equations of motion in Newtonian or Hamiltonian form, respectively. Together they form a **q-classical phase space** $\mathbb{M}$ derived from the structure and dynamics of the quantum system that includes all the primitive microscopic variables. **We have therefore recovered CM as a structure inside QM.** Non-equilibrium statistical CM can be defined[18] on top of that phase space $\mathbb{M}$ of the q-classical variables with its dynamics [77,97,12].

Note that the q-classical systems are derived from the fully quantum description of macroscopic systems, which is distinct from, but compatible with, the hybrid quantum-classical systems considered by Elze [38].

We obtained the following construction as the realist formulation of QM in four layers:

$$\begin{aligned} &|\Phi\rangle \in \mathbb{H} \\ &\sigma : \mathcal{B}(\mathbb{H}) \to \mathbb{R} : S \mapsto \sigma(S) \\ &(Q,P) = (m_{\boldsymbol{A}}(\sigma), m_{\boldsymbol{B}}(\sigma)) \in \mathbb{M} \\ &\sigma' : \mathcal{B}(\mathbb{M}) \to \mathbb{R} : S \mapsto \sigma'(S) \end{aligned} \tag{2.22}$$

consisting of deterministically evolving kets in $\mathbb{H}$, a quantum statistical state σ that is a probability measure on $\mathbb{H}$, a set of q-classical variables (Q, P) defined with that statistical state σ that form a classical phase space $\mathbb{M}$, and a classical statistical state σ' that is a probability measure on the q-classical phase space $\mathbb{M}$.

3 Measurement

We apply both the empiricist formulation [60] of the measurement process according to ABN[19] and the realist formulation developed in the paper to a concrete experiment. We have two goals with this presentation and associated analysis:

The first goal is the same as that of ABN and is to show that measurement is a complex process. Too often the measurement problem is presented in an overly abstract way, with kets or density operators that represent very large numbers of degrees of freedom being manipulated through a small sequence of steps, while the countless complex processes taking place at various stages in the measurement process are ignored [93]. Fortunately, the modern discussion includes the process of decoherence to counter this oversimplification [91].

The second goal is to show that a careful and precise deduction from the foundation assumption of kets evolving under the SE, viewed in the Heisenberg-Dirac ontology,[20] does meet up precisely with the inductive empiricist description of quantum phenomena, as given, for example, by ABN. In other words, the realist formulation provides an explanation of what happens **inside** the black box that is QM, the **outside** of which is carefully described by the empiricist formulation of QM [60].

To illustrate the role of probability, the place of classical systems, and the measurement process in QM, we consider the famous double-slit thought experiment that was carried out as a real experiment in 1989 by Tonomura, Endo, Matsuda, Kawasaki, and Exawa [86].

[18] The procedure is summarized in Appendix B.

[19] The ABN description of the measurement process is summarized in Appendix C.2.

[20] The definition is given and discussed in Appendix C.3.

1. **Electron beam** A beam of electrons, accelerated in an electron microscope by an electrostatic potential of 50 kV, is sent through a biprism, which consists of two parallel grounded plates with a thin filament on a positive potential placed between them. 50 kV electrons move at approximately half the speed of light. The distance from the source to the detector is about 1.5 m, which is traversed in a tenth of a microsecond. The electron arrival rate in the experiment is 1,000 electrons per second on the entire 113 mm^2 detector surface. With about one electron injected into the system every 1,000 microseconds, it is clear that the electrons do not interact with each other during their flight through the electron microscope to the detector.
2. **Detector** The detector surface, which has a diameter of 12 mm, is a combination of a fluorescent film and a photon-counting image acquisition system. When a 50 kV electron hits the fluorescent screen, about 500 photons are produced at the location of impact. Touching the fluorescent film is an array of optical fibers to guide the photons to an amplification and position recording system. The photons produce secondary electrons that are accelerated. An electrostatic lens produces a point image on the upper layer of a multichannel plate, where the number of electrons is multiplied and the position recorded in a computer. The accumulated image is then shown on the display monitor.
3. **Observation** The electron de Broglie wave length is 0.54 nm. The distance between two interference fringes is $d = 70$ μm. The interference fringes are magnified 2,000 times by two projector lenses onto the detector. The enlarged fringe spacing is then 1.4 mm. The interference pattern can be generated in approximately 20 minutes. The authors show [86] a sequence of photographs (reproduced in Ref. [44]) of the buildup of the interference pattern: A few random light dots on a black background are visible in the initial photograph, with subsequent photographs showing increasing numbers of light dots; five fringes are shown in the final photograph after about 70,000 electrons have been recorded, about 14,000 electrons per fringe.

The empiricist approach to the measurement process [60] must logically start from the data observed in an experiment, which means the process must include the recording of the data, often referred to as the "pointer variable." In general this is where the wave function is considered to collapse [91]. The description of all the processes up to that point are treated as described by the SE [54] and are often called "premeasurement" [64,23]. In the above double-slit experiment, all processes up to the recording in computer memory can be considered premeasurement. The description following ABN in Sect. 3.1 describes the full measurement in this sense; the description using the realist view in Sect. 3.3 focuses on the interaction between the electron wave function and the fluorescent film because it creates the initial irreversible amplification by creating 500 photons at one spot on the film. This is part of premeasurement and follows van Kampen's description [54] of the measurement process. The rest of the process can be described similarly using the realist formalism, because there is no wave function collapse, just more deterministic evolution according to q-classical dynamics as defined in Sect. 2.3.

In Sect. 3.1 we describe the experiment using the language and formalism of ABN. Next, in Sect. 3.2 we analyze the role of the position observable for the electron in this experiment. In Sect. 3.3, we describe the process using the realist formulation of statistical QM developed in this paper, Sect. 2. Then a derivation of Born's rule follows in Sect. 3.4.

3.1 Empiricist description

We now describe the double-slit experiment by applying the work by ABN [3,4], summarized in Appendix C.2, to show how the unique outcomes that are probabilistically recorded follow from the dynamics governed by the LvNE Eq. (B.10). The Copenhagen postulate of a second way, projection or collapse, for the ket to evolve is therefore not necessary [3, pp. 8,140]. The experimental preparations and observations take place by controlling macroscopic variables [3, p. 10].

3.1.1 Electron beam

The electron beam is characterized by a statistical state $\boldsymbol{D}^S$ Eq. (B.8) for the electron system S. The mean of that statistical state is the wave function of the electrons in the beam; the variance is very small and negligible for all practical purposes. By careful control of the experiment, the number of electrons flowing through the electron microscope is about 1,000 per second. Thus the dynamics of the electron can be described by focusing on the wave function until it hits the detector screen. As the electron interacts with the detector, only a description in terms of the statistical operator $\boldsymbol{D}$ of the electron system S and the measurement apparatus M will work [3].

3.1.2 Detector

To describe the detection process, the fluorescent film on the 12 mm diameter detector M is viewed as an array of mini detectors, called "grains" by ABN, $M(x, y)$ that are assemblies of fluorescent molecules located at coordinates $(x, y, z = 0)$, assuming the electron beam flows downward along the z-axis from the source at $z = h = 1.5$ m to the detector at $z = 0$. Each mini detector has a observable $\boldsymbol{W}(x, y)$, indexed by the x, y coordinates of the position in the $z = 0$ plane of the detector. The observable indicates whether the mini detector has fired a bunch of photons. The observable is defined as the average over the molecules in the mini detector of the amplitude of the excited state that will emit a photon when the molecule transitions back down. The grains $M(x, y)$ correspond to the magnet M in the Curie-Weiss model studied by ABN and the observable corresponds to the magnetization. Applying their model the mini detectors are the systems with a coherent dynamics that can absorb the electron energy and record the absorption event by emitting 500 photons almost simultaneously in a flash.

Given the very large number of fluorescent molecules in the screen, the number of mini detectors is also very large. Assume that the fluorescent molecule in the film has about 30 atoms and has a diameter of 10 nm. It therefore has a cross section of 80 nm^2. With a 12 mm diameter, the screen has $\pi 6^2$ mm^2/ 80 nm^2 $\simeq$ 10^{12} fluorescent molecules. Thus there are $2{\times}10^9$ $\simeq$ $10^{12}/500$ mini detectors, each with a surface area of $500{\times}80$ nm^2 $\simeq 0.04\,\mu$m^2.

The number 500 of fluorescent molecules in any mini detector is itself defined by the dynamics of the combined system and is not an exact number for every detection event. It is determined by the number of photons that can be created by the energy of a 50 kV electron. Since each photon is created by the transition in one fluorescent molecule, one electron impact event must involve 500 molecules that interact in a quantum mechanically coherent way. It is in this sense that the grains are defined dynamically as part of the electron impact.

Given that a typical optical fiber has a diameter of 10 μm and a cross section of $\pi 5^2\,\mu$m^2 $\simeq 80\,\mu$m^2, a fiber will capture the photons from approximately 80 μm$^2/0.04\,\mu$m^2 $\simeq 2,000$ mini detectors.

Because each mini detector consists of 500 molecules, each with about 30 atoms, the detector variable $\boldsymbol{W}(x, y)$ is determined by about 15,000 atomic observables so that it has a probability distribution with a width that is narrower than the atomic observables by a factor of $\sqrt{15,000} \simeq 122$, i.e. two orders of magnitude narrower. This meets the requirements on N to make the mini detector sufficiently large to be a pointer system as defined by ABN [3, p. 52].

The statistical state $\boldsymbol{D}^M$ of the complete detector screen is a product of the statistical states $\boldsymbol{D}^{M(x,y)}$ of all mini detectors. While the mini detectors may interact at the edges, there is no persistent quantum coherence across large numbers of mini detectors. The statistical state $\boldsymbol{D}^{M(x,y)}$ is very different from that of the carefully prepared beam. It is not narrowly focused on a set of physically similar wave functions, but instead covers a broad class of wave functions and all their superpositions. The statistical state $\boldsymbol{D}^M$ of the whole detector is then constructed as a product of the statistical states of the mini detectors, possibly with the addition of some corrections to allow for entanglement between neighboring mini detectors.

To describe the transition of the mini detector from initial state to final state where 500 photons have been omitted, we use a basis for the grains constructed as a tensor product from the basis for each molecule, with minor modifications that can be computed by perturbation theory to take into account the effect of the interaction, including chemical bonding, between the fluorescent molecules. That basis has a state with all molecules in their lowest electronic state and numerous states with various levels of electronic excitations. We are interested in the first electronic excited state in each molecule, as the transition from that state to the molecular ground state will generate one photon. The interaction with the electron generates an excited state for the grain with about 500 molecules in their excited state. This excited state will then decay with the emission of 500 photons.

3.1.3 Observation

With the above identification of the physical subsystems in the double-slit experiment, we now follow the description of the measurement process given by ABN.

The **preparation** consists of the electron interacting with all grains in randomly distributed statistical operators $\boldsymbol{D}^{M(x,y)}$ as described by the LvNE. The impact of the electron stimulates the molecules in the ensemble of grains to excited states as a quantum mechanical process involving the electron and about 500 molecules. This puts the mini detectors in a metastable state.

The next part of the evolution is the **initial truncation** where one of the mini detectors in the ensemble proceeds to emit 500 photons. Here the off-diagonal terms in the grain basis of the statistical operators of the grains evolve to zero and $\boldsymbol{D}^{M(x,y)}$ becomes diago-

nal. The intensity of the electron beam is such that this happens only for one out of the ensemble of 2×10^9 mini detectors at a time (with very high probability).

Irreversible truncation of the off-diagonal elements in the statistical operators in the grain basis described above results from further interaction among the molecules in the grain and of the photons with the fiber optic that ensure that the process is irreversible and that there is no recurrence of the off-diagonal blocks.

The photons travel through the optical wires to the recording system which generate secondary electrons that get amplified and then recorded as described above. **Registration** takes place where correlation is built up between the electron system S, the ensemble of mini detectors $M(x,y)$, and the recorder system. Because of the experimental setup, where only one electron is present at any one time, the ensemble will be such that only one mini detector emits 500 photons at any one time and only its secondary electrons are amplified and recorded in the computer memory.

Further interactions ensure **sub-ensemble relaxation** so that each run can be associated with a unique outcome of the position of the flash being registered.

Finally, the **reduction** takes place with the 500 molecules returning to the ground state leaving 500 photons to be caught in the optical fibers, which initiates secondary electrons to be emitted and amplified, producing a current that is recorded in a computer.

Now the measurement of a single electron passing through the microscope is complete, and one dot has been added to the display.

3.2 Position observable: particles or fields?

In the empiricist view [60], all observables are on equal footing: They produce a spectrum and a probability distribution for finding values in that spectrum. If the electrons are described by the non-relativistic SE with the position operator $\boldsymbol{r} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ defining the Schrödinger picture, the evolution of the ket and its wave function in that representation can be computed. Upon detection, a single isolated flash is observed with the probability given by Born's rule as a consequence of the LvNE evolution of the interacting systems that were described in Sect. 3.1.

Now we want to describe the same process in a realist way, starting from the deterministic evolution of kets as discussed in Sect. 2.1 for the underlying quantum systems, which includes the electrons in the beam of the electron microscope (as well as all other electrons in the components of the measurement apparatus).

Then the question arises: What happens to that wave function during the interaction with the macroscopic instrument as a quantum system? In particular, what happens to the wave function during and after the interaction with the field of mini detectors $M(x,y)$ when one of them emits 500 photons and all the others do not?

If we hold the realist notion that the ket is the electron, then the ket must now be localized around the position (x_1, y_1) of that one mini detector $M(x_1, y_1)$ that sent off 500 photons. In other words, the wave function has collapsed. The transformation of the ket from just before that interaction to just after that interaction **cannot be described by the SE**. What are the options to proceed in the development of a theory for quantum phenomena?

1. The traditional conclusion, the empiricist view, is to declare this detail as being inside the black box; we do not know and we do not have to know to be able to do physics.
2. The realist view taken in this paper follows the intuition of the young Heisenberg [50,42] to let the formalism be the guide, and **rejects the position operator as an admissible observable** even in non-relativistic QM.

The need for considering position an observable has a natural alternative: Consider the position as the coordinate in space of the event under consideration, instead of taking the position as an observable of the system involved in the event. In the case of the double-slit experiment, the electron creates an event and the detector records the event with the wiring of the detector allowing tagging the event with the coordinates in space of the event. Similarly, there has been an interest in considering time as an observable [22, §III.4.2], instead of as a coordinate of an event. In CM the two views are trivially equivalent, since one only measures a few numbers; in QM, however, there is a big difference as a coordinate is a number and an observable is a Hermitian operator.

Within the context of non-relativistic QM, we can accept instantaneous changes across infinite distances in space. In CM this never poses a conceptual problem and the only change that ever needs to be made to a classical non-relativistic theory to make it relativistic and Lorentz covariant is to make the speed of propagation finite. Careful analysis of the problem of nonlocality and collapse of the wave function by d'Espagnat [27, §8.3], later refined and confirmed by Aharonov and Albert [1,2], shows that the concept of a wave function and the process of its collapse cannot be described in a consistent Lorentz covariant way. This is further substantiated by the fact that no good relativistic position operator seems to exist [85].

If we assume that all physics is described in terms of local quantum fields, with all kets $|\Psi, x, y, z, t\rangle$ and observable operators $\boldsymbol{\varphi}_{x,y,z}$ labeled by space coordinates (x, y, z) evolving in time t, then the Heisenberg-Dirac[21] ontology offers a clean picture of kets evolving in Hilbert space under the SE at all times for all processes, fully accessible by theory and computation, possibly somewhat hidden from experiment.

There is an equivalent formulation of non-relativistic many-body QM that takes the form of a field theory. Consider the SE for N particles with mass m moving in an external potential $U(\mathbf{r})$ and interacting through a potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ [83, p. 11]

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} = &\sum_{j=1}^{N}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{r}_i)\right)\Psi \\ &+ \sum_{j=1}^{N}\sum_{k=1}^{j-1} V\left(|\mathbf{r}_j - \mathbf{r}_k|\right)\Psi. \end{aligned} \tag{3.1}$$

One introduces the field operator $\boldsymbol{a}(\mathbf{r})$ and its Hermitian conjugate $\boldsymbol{a}^\dagger(\mathbf{r})$ satisfying the field commutation relations

$$\begin{aligned} [\boldsymbol{a}(\mathbf{r}), \boldsymbol{a}(\mathbf{r}')] &= 0 \\ [\boldsymbol{a}^\dagger(\mathbf{r}), \boldsymbol{a}^\dagger(\mathbf{r}')] &= 0 \\ [\boldsymbol{a}(\mathbf{r}), \boldsymbol{a}^\dagger(\mathbf{r}')] &= \delta^3(\mathbf{r} - \mathbf{r}'). \end{aligned} \tag{3.2}$$

Define the Hamiltonian operator

$$\begin{aligned} \boldsymbol{H} = &\int \boldsymbol{a}^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r})\right)\boldsymbol{a}(\mathbf{r})\mathrm{d}^3r + \\ &+ \frac{1}{2}\int\int V(|\mathbf{r} - \mathbf{r}'|)\boldsymbol{a}^\dagger(\mathbf{r})\boldsymbol{a}^\dagger(\mathbf{r}')\boldsymbol{a}(\mathbf{r}')\boldsymbol{a}(\mathbf{r})\mathrm{d}^3r\mathrm{d}^3r'. \end{aligned} \tag{3.3}$$

Then the time-dependent ket

$$\begin{aligned} |\Psi, t\rangle = &\int \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) \\ &\boldsymbol{a}^\dagger(\mathbf{r}_1)\ldots\boldsymbol{a}^\dagger(\mathbf{r}_N)\,|0\rangle\,\mathrm{d}^3r_1\ldots\mathrm{d}^3r_N, \end{aligned} \tag{3.4}$$

where $\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)$ is the many particle wave function in the Schrödinger representation and $|0\rangle$ is the zero-particle vacuum, can be shown to satisfy the abstract SE Eq. (A.11) if Ψ satisfies Eq. (3.1). This equivalent representation is known as the "second quantization formulation" [31, §5.4] [45, p. 46] [27, p. 28] and is considered a very efficient framework for computations in many-body QM [69, §6.3].

In this representation of non-relativistic QM, there is no position observable with a Hermitian operator, only a field of operators labeled by position as a parameter that gives the coordinates in space where the wave function of the field is considered. Now space and time are treated the same way. Even though the SE is still non-relativistic, the formalism is ready for the transition to a relativistic treatment.

It follows that in the realist approach presented here, all systems must be described as quantum fields, the interactions between mini detectors must be described this way as well and there is no concern about describing large arrays of mini detectors. In particular, the no-interaction theorem of Currie [26,9] for systems of particles described can be viewed as further motivation to avoid the position of particles as an observable.

Any questions of what happens when the ket of a system evolves to cover a macroscopic region and then, as the result of interactions with a macroscopic array of mini detectors, produces a localized emission of 500 photons, can properly be formulated as a question regarding the ket $|\Psi, \mathbf{r}, t\rangle$ of the quantum field with the associated wave function $\Psi(\mathbf{r}, t)$. The non-relativistic SE applied to the quantum field still predicts instantaneous changes across large macroscopic regions, because it is a non-relativistic equation. The correct answer to such questions on locality and propagation can only be obtained by considering and solving the relativistic equation.

After this analysis, we can return to the question of what happens to the ket and wave function of the electron after the interaction with all mini detectors. The ket of the electron quantum field evolves through the double-slit setup and interacts with all detectors. The quantum field ket has a wave function representation on the field of operators $\boldsymbol{\varphi}_{x,y,z}$ and the local field operators interact with the mini detectors. One of the mini detectors interacts in such a way that 500 photons are emitted and the quantum field continues to evolve locally around that mini detector to reflect the result of the interaction. This includes the loss of energy transferred to the photons. In general, the wave functions manipulated in the second quantization formalism are chosen to be eigenstates of the number operator. And this brings us right back to the concern about collapse of the wave function, because now there is a constraint that the total number of electrons is one across a macroscopic region! However, the answer should be clear: Interacting quantum fields may very well never be in an eigenstate of the free-field number operator, hence there is no fundamental constraint across a macroscopic region. The fact that only one mini-detector fires at a time is controlled by the condition that the intensity of the generated electron quantum field is sufficiently low. There is a small probability that two mini detec-

[21] Definition in Appendix C.3.

tors fire 500 photons at exactly the same time, since the interactions at each mini detector are independent.

We take inspiration from axiomatic [95,84,13] and algebraic [5,45] quantum field theory to modify the formulation of deterministic QM in Sect. 2.1 to include the restriction that **position and time are excluded as observables**, even in non-relativistic QM. In other words, all physical systems must be described in terms of local observables of quantum fields, preferably relativistic quantum fields.

In this framework, all particles are to be considered as excitations (quanta) in suitable quantum fields so that space and time are handled consistently and so that the field dynamics is able to correctly describe the propagation of any and all effects, interactions, and correlations in spacetime.

We do not address the mathematical issues associated with the explicit construction of the Hilbert space for quantum fields or the solution of the SE for their evolution. We take the success of relativistic quantum field theory in computing any and all desired observables [79] as an indication that accepting the postulate is workable.

3.3 Realist explanation

With this clarification, we can now describe the double-slit experiment from the realist perspective using the formalism developed in this paper using the statistical state defined in Sect. 2.2 and the notion of q-classical systems introduced in Sect. 2.3. The goal is to show that this view is consistent with experimental observations. As the empiricist description in Sect. 3.1 digs down from experiment to foundations, the realist description climbs up from foundations toward experiment. We will see that the two approaches meet in the middle and line up perfectly.

As noted, the statistical states σ are manipulated by controlling a small number of macroscopic collective coordinates.

3.3.1 Electron beam

We look at the electrons as excitations in the electron quantum field, so that a statistical state does not necessarily have a sharp number of electron excitations in the field. By careful control of the experiment, the number of quanta flowing through the electron microscope is about 1,000 per second. But the quantum field ket is not an eigenvector of the number operator with eigenvalue one.

The electron beam in the double-slit experiment is characterized by a very narrow statistical state σ^S of the electron system S, such as the high-precision state described in Section 2.2.3. The mean of that statistical state is the wave function of the electron that is the focus of the traditional quantum description of the experiment. Let that wave function $\Phi(x,y,z)$ describe the electron, so that the non-relativistic quantum field is given by Eq. (3.4)

$$|\Phi\rangle = \int_{\mathbb{H}} |\Psi\rangle\sigma(\mathrm{d}|\Psi\rangle) \tag{3.5}$$

It is well known that this wave function covers a macroscopic region of space after passing through the double slit [40] and that it shows the diffraction pattern in the $x-y$ plane. A computation of the q-variance Eq. (2.14) of the observables $\boldsymbol{x}$ and $\boldsymbol{y}$ in this ket show it not to be small. Because the statistical state σ is a high-precision statistical state (§2.2.3), the mean and variance of these variables in that statistical state σ are the same as the q-expectation and q-variance, respectively, of its mean ket $|\Psi\rangle$. Hence, the variances Eq. (2.16) are large and the means $m_{\boldsymbol{x}}(\sigma)$ and $m_{\boldsymbol{y}}(\sigma)$ Eq. (2.15) for the observables are not q-classical variables.

3.3.2 Detector

The analysis of the detector follows closely the discussion given in Section 3.1.2, with the main difference that the statistical state is now described as a measure σ^M on the quantum mechanical phase space $\mathbb{H}$. The measure for the detector is constructed, as described in Section 2.2.4, by assembling the measures $\sigma^{M(x,y)}$ of the mini detectors. As we saw in Section 3.1.2, a mini detector is a physical system with about 15,000 microscopic degrees of freedom and the detector contains 2×10^9 mini detectors. It is impossible, in practice, to characterize the ket of such a system or even specify a high-precision statistical state for it. Rather it is described by a large-system statistical state (§2.2.4), which is a probability measure with a large variance. The mean ket of σ^M will have a probability similar to the kets of many other kets with significant variation in the kets across all mini detectors; including kets for mini detectors in which various fluorescent molecules are in excited states. Kets that are superpositions and entanglements involving several molecules, and even superpositions and entanglements involving all molecules in the mini detector, also carry significant probability in the individual statistical states $\sigma^{M(x,y)}$ and the total state σ^M, respectively.

The interaction with the 50 keV electron beam to produce 500 photons is what determines the size of each mini detector. The mini detectors consist of a number of fluorescent molecules that are sufficiently close together

to make the quantum dynamics under the electron impact coherent; no smaller part can be considered to have dynamics that can be treated to good approximation in isolation from the surroundings. The collection of mini detectors forms a grid $G = \{(x_i, y_i) | i = 1, \ldots, I\}$ with $I \simeq 2 \times 10^9$ from the estimate in Section 3.1.2. The statistical state σ^M of the whole detector is then constructed as a product

$$\sigma^M = \bigotimes_{(x,y)\in G} \sigma^{M(x,y)} \tag{3.6}$$

of the statistical states of the mini detectors.

For the physical characterization of the mini detectors, we refer the reader back to Section 3.1.2. We denote by $\Theta(q_1, \ldots, q_N)$ the second-quantization wave function of all the nuclei and electrons in the molecules in one mini detector. Then we compute the mean Eq. (2.15) of the detector variable $\boldsymbol{W}(x, y)$

$$\begin{aligned} m_{\boldsymbol{W}(x,y)}(\sigma) &= \int_{\mathbb{H}} v_{\boldsymbol{W}(x,y)}(|\Theta\rangle)\sigma(\mathrm{d}|\Theta\rangle) \\ &= \int_{\mathbb{H}} \langle\Theta|\boldsymbol{W}(x,y)|\Theta\rangle\sigma(\mathrm{d}|\Theta\rangle) \end{aligned} \tag{3.7}$$

and the variance Eq. (2.16)

$$\begin{aligned} \Delta_{\boldsymbol{W}(x,y)}(\sigma) &= \left(\int_{\mathbb{H}} w^2_{\boldsymbol{W}(x,y)}(|\Theta\rangle)\sigma(\mathrm{d}|\Theta\rangle) \right)^{\frac{1}{2}} \\ &= \Bigg(\int_{\mathbb{H}} \Big(\langle\Theta|\boldsymbol{W}(x,y)^2|\Theta\rangle \\ &\qquad - \langle\Theta|\boldsymbol{W}(x,y)|\Theta\rangle^2 \Big) \sigma(\mathrm{d}|\Theta\rangle) \Bigg)^{\frac{1}{2}} \end{aligned} \tag{3.8}$$

with $\mathbb{H} = \mathbb{H}^{M(x,y)}$ the Hilbert space of detector $M(x, y)$.

We conclude, as before, that the mini detector observable $\boldsymbol{W}(x, y)$ is determined by about 15,000 atomic observables, so that it has a variance that is narrower than the atomic observables by two orders of magnitude. It follows that the variance $\Delta_{\boldsymbol{W}(x,y)}(\sigma^{M(x,y)})$ is sufficiently small. By Definition 2.4, this implies that the mean $m_{\boldsymbol{W}(x,y)}(\sigma^{M(x,y)})$ is a q-classical variable $Q_{x,y}$.

The q-classical coordinate $Q_{x,y}$ measures the occupation of the excited states in the mini detector $M(x, y)$, which will result in emissions of photons. We associate a rate of change $P_{x,y} = \dot{Q}_{x,y}$ as q-classical conjugate coordinate to $Q_{x,y}$. The q-classical variables $(Q_{x,y}, P_{x,y})$ can therefore be used to define a classical phase space $\mathbb{M}(x, y)$ for each mini detector $M(x, y)$ in the statistical state $\sigma^{M(x,y)}$. The dynamics in that phase space follows from the SE by Ehrenfest's Theorem 2.1. We denote the collection of all mini detector q-classical variables by $(Q, P) = ((Q_{x,y}, P_{x,y})_{(x,y)\in G}$ and the resulting q-classical phase space by $\mathbb{M}$.

3.3.3 Observation

The statistical state of the system of the electron beam being measured by the detector is given at the initial time $t = 0$ by the measure $\sigma_0^{SM} = \sigma^S \otimes \sigma^M$ on the Hilbert space $\mathbb{H} = \mathbb{H}^S \otimes \mathbb{H}^M$ and its evolution Eq. (2.6) is given by the Schrödinger flow Eq. (2.5) to yield σ_t^{SM}. The evolution immediately destroys the product form of the measure and gives probability to numerous kets that are superpositions and entanglements of kets of the beam system and the detector system.

The beam of electrons is a microscopic quantum system that interacts, because of the magnification in the electron microscope, with all the mini detectors in the detector. Because the system is described as a system of interacting quantum fields, the SE describes the evolution of the kets of the interacting system by the motion in Hilbert space $\mathbb{H}$ driven by interactions local in space and in time. Each initial ket in the statistical state of the beam couples with all mini detectors, and it evolves as part of the Schrödinger flow indefinitely.

Because the initial statistical state $\sigma^S \otimes \sigma^{M(x,y)}$ has probability on a very large number of kets, the relevant part (i.e. with nonzero weight) of the Schrödinger flow has a wide range of outcomes. Some kets show the electron excitation just shoot through the fluorescent film. Others bring a few molecules to the excited state that then decay, each sending out a few photons, but too few to be recorded. Still others have an evolution where many molecules get excited in a coherent way so that a large number of photons is emitted taking the majority of the 50 keV energy carried by the electron excitation. For some mini detectors, there is a sufficiently large fraction of kets of this latter type, so that the average over the statistical state $\sigma^{M(x,y)}$ as a q-classical variable shows emission of many photons, in numbers large enough for amplification, thus creating a recorded event as indicated by the evolution of the q-classical coordinates $(Q_{x,y}, P_{x,y})$. By definition q-classical variables have low variance, i.e. they are dispersionless and hence like the variables in classical physics [51].

In the realist description, in contrast with the empiricist description following ABN in Sect. 3.1, there is no need to follow the amplification of a detection signal all the way to include the photons generating an electrical current which then results in the event being written in the computer memory. The emission of 500 photons by the randomly selected mini detector is sufficient to establish the measurement as an event; the rest is just q-classical dynamics.

It may be useful to recollect, at this juncture, where the probabilistic nature of QM enters in the realist explanation. The kets of the electron coming in and those

of all the molecules in every mini detector all evolve deterministically (§2.1) governed by the SE throughout the entire process, from before the measurement event until after it. The mini detectors are in kets that are randomly distributed according to the probability measure $\sigma^{M(x,y)}$ on the Hilbert space $\mathbb{H}^{M(x,y)}$ so that some of them are in the correct ket to coherently interact with the incoming ket of the electron, which is narrowly distributed by probability measure σ^S, and produce 500 photons, instead of none or an undetectable number. That is the randomness in all quantum processes. Then the q-classical variables, driven by the underlying SE and determined by the statistical state, present the classical world, which experiments record.

In the realist narrative, there is no need to distinguish the premeasurement [64] stage, as there never is a collapse of the wave function. The variables that are observed and recorded in experiments are associated with the statistical state σ as q-classical variables, they are not associated with the underlying ket in the Hilbert space $\mathbb{H}$; the ket never collapses, it changes, evolves, and keeps going.

3.4 Born's rule

From the analysis of the double-slit experiment using both the empiricist and the realist views, we obtain the following corollary of Ehrenfest's Theorem 2.1.

Corollary 1 (Measurement) *The Schrödinger flow of an observed quantum system and a quantum apparatus results in the evolution of a detector in a measurement process being described by the evolution of the q-classical system derived from the statistical state of the quantum system.*

Proof The detector behaves like a statistical ensemble of mini detectors. The initial state is given by the electron beam and the random initial statistical state of each mini detector.

Each mini detector, and hence the whole detector, has created a definite outcome for each detection event in a q-classical system, defined by the statistical state of the macroscopic quantum system that is the measurement apparatus.

The quantum statistical state σ_t^{SM} defines a probability density $\rho_t(Q,P)$, and hence a probability measure σ'_t, on the q-classical phase space $\mathbb{M}$ with the q-classical variables (Q,P). The evolution σ'_t of this measure fully characterizes the dynamics of the detector as the q-classical statistical state of a q-classical system with coordinates (Q,P) defined by the quantum statistical state σ_t^{SM}.

The q-classical state (Q,P) of the q-classical apparatus associated with the statistical state σ_t^{SM} of the beam-apparatus as a quantum system is the recording of the observed event.

After the measurement process, the statistical state of both the microscopic quantum system being measured (the electron) and the macroscopic system that is the apparatus (the molecules in the mini detectors, the photons, and recording devices) continue to evolve as dictated by the Schrödinger flow. □

The mini detectors are the natural smallest units in the macroscopic detector system. Their dynamics is coherent in that superpositions and phases play a role in the evolution of the electron and the molecules in the mini detector to allow extraction of the energy from the electron by a sufficient number of fluorescent molecules so that each one can send out a photon.

Alternatively, we can think of mini detectors as the largest units into which the macroscopic detector can be divided so that it can still be described as a collection of parts each evolving mostly independently so that the q-classical variables $(Q_{x,y}, P_{x,y})$ for different (x,y) are statistically independent variables.

In the discussion so far we have formulated all statements in terms of kets which are elements, vectors, in Hilbert space. The Hilbert space can be given a realization once a complete set of commuting observables is chosen. Then the kets are represented by wave functions: complex-valued functions on the combined spectrum of the set of chosen observables. For a single-degree-of-freedom system with observable q, wave functions are complex functions $\Psi(q) = \langle q|\Psi\rangle$ of q.

We have further formulated all systems as fields so that all observables are indexed or labeled by spatial position, for example for a scalar field $\boldsymbol{\varphi}_{\mathbf{r}} = \boldsymbol{\varphi}_{x,y,z}$. The same holds for the kets: $|\Psi, \mathbf{r}\rangle$.

We are now in a position to derive Born's probability rule. Following Feynman and Hibbs [40, p. 96] that every measurement is realized as a set of observations labeled by space and time, we formulate Born's rule using position and time. All observables in quantum mechanics, even internal ones like spin and field strengths, are observed in practice by using the SE to connect the possible observable values to specific events at definite locations and times.

In the double-slit experiment we have established how events that are localized at a macroscopic scale are produced by the dynamics in QM. From the observations, we can reconstruct a phenomenological representation that includes Born's probability interpretation of the wave function as follows.

Theorem 3.1 (Born's rule) *Given a setup consisting of a macroscopic measurement apparatus and a microscopic observed system in the statistical state σ_t with mean ket $|\Psi,t\rangle$ such that*

1. *there is a collection of mini detectors located on a grid of positions, and*
2. *the mean ket $|\Psi,t\rangle$ has evolved to cover a macroscopic area containing mini detectors,*

then a phenomenological wave function $\Psi_{\text{phen}}(x,y,z,t)$ can be constructed from the mean ket $|\Psi,t\rangle$ such that the probability density for the apparatus to register a detection of the microscopic system in position (x,y,z) at time t is given by

$$p(x,y,z,t) = |\Psi_{\text{phen}}(x,y,z,t)|^2, \tag{3.9}$$

the modulus squared of the phenomenological wave function.

Proof Consider the position operator as the multiplicative operator $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ acting in the phenomenological Hilbert space $\mathbb{H}_{\text{phen}} = L^2(\mathbb{R}^3,\lambda,\mathbb{C})$ of complex-valued, square-integrable functions with λ Lebesgue measure on $\mathbb{R}^3$.

For the sake of keeping the notation from becoming a distraction, we consider the electrons in our double-slit experiment as spinless, so that the electron quantum field is a scalar field. We now associate with the ket of the electron quantum field in and near the detector at $z = 0$ the phenomenological non-relativistic wave function $\Psi_{\text{phen}}(x,y,z)$ that satisfies the following condition

$$|\Psi_{\text{phen}}(x,y,z,t)|^2 = \int_{\mathbb{R}} \left|\langle\varphi_{x,y,z}|\Psi,t\rangle\right|^2 \mathrm{d}\varphi_{x,y,z} \tag{3.10}$$

Here $|\Psi,t\rangle$ is the mean ket for the electron quantum field, $\varphi_{x,y,z}$ is the spectral variable for the field operator $\boldsymbol{\varphi}_{x,y,z}$, and Ψ_{phen} is the phenomenological wave function that is an element of $\mathbb{H}_{\text{phen}}$.

The solution of this problem does not uniquely fix the phase of the phenomenological wave function, but that is not a problem because the granularity of the mini detectors is defined precisely by the stipulation that different mini detectors evolve (mostly) independently and no phase coherence between them exists.

Using the formalism of second quantization, Section 3.2, in the non-relativistic approximation of a low intensity beam with only a single electron in flight at all times, the phenomenological wave function is the wave function in the single-quantum sector of Fock space of the electron field. The claim follows. □

It follows from the proof that within the realist formalism presented here the operational meaning of the Born rule is only valid for kets $|\Psi,t\rangle$ that have evolved to cover regions of macroscopic extent in space. It is not valid at or below the nanoscale, since there are no detectors that can operate at that scale. For example, the Born rule does not hold inside an atom.

4 Summary and outlook

In this paper, a realist formulation of QM is presented to approach the measurement problem. The foundation principles are that kets are the basic building blocks of QM and that their evolution is governed at all time and forever by the SE (§2.1). The probabilities, inextricably connected with quantum phenomena, are introduced by placing a probability measure on the Hilbert space of kets (§2.2). The approach goes by way of defining an explicit construction of q-classical systems derived from quantum systems (§2.3), which are shown to satisfy the laws of CM and therefore can be identified as mapping all of CM into QM, formally showing that CM is a part of QM.

If we accept the plausible conclusion that the description of the ideal measurement process developed by ABN [3,4] can be applied to all measurement processed, then the realist explanation presented in this paper obtains the same results, just in a different way. The contribution of the paper is to show that a realist constructive formulation of QM can be given that starts from the Schrödinger equation, and the framework in which it is formulated, and derives the complete quantum phenomenology. It provides the complement to the empiricist activity of induction from experiment to principles by providing a complete deduction from the dynamical law of QM given in 1926.

The realist narrative turns the traditional view of QM on its head: Instead of QM being inherently probabilistic, it is the classical systems derived from quantum systems that are intrinsically statistical. To be more precise, whereas the empiricist description holds QM as inherently probabilistic, the realist explanation makes kets deterministically evolving components; the collective variables that constitute q-classical systems, while governed by deterministic classical laws, are all statistical variables.

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A Brief formulation of quantum mechanics

For ease of reference and consistency of conventions, nomenclature, and notation, we give a brief overview of the formalism of QM.

In CM, a degree of freedom is described by a coordinate q and its velocity $\dot{q}$ (Lagrangian formalism) or its conjugate momentum p (Hamiltonian formalism). With the set of degrees of freedom for a system specified, one selects the range of values, called the spectrum, of one coordinate in each pair, typically q, and assembles them into the **configuration space**. For example, a system consisting of N degrees of freedom, with each coordinate having the real numbers as spectrum, has the configuration space

$$\mathbb{F} = \mathbb{R}^N. \tag{A.1}$$

The **phase space** in QM is then defined as the Hilbert space of complex-valued functions on the configuration space

$$\mathbb{H} = L^2(\mathbb{F}, \lambda, \mathbb{C}) = \otimes^N L^2(\mathbb{R}, \lambda, \mathbb{C}). \tag{A.2}$$

Here λ denotes the Lebesgue measure on $\mathbb{F}$ and $\mathbb{R}$, respectively. The second form of the phase space in Eq. (A.2) shows its structure as a composition, tensor product, of the N degrees of freedom [61], similar to the decomposition of the phase space in CM as

$$\mathbb{H} = \oplus^N \mathbb{R}^2, \tag{A.3}$$

where $\mathbb{R}^2$ is a two-dimensional phase space of a single degree of freedom with the symplectic form

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.4}$$

This form of phase space is called the **Schrödinger representation**. The functions on configuration space that are the elements of phase space are called **wave functions**. The **state** of a system in QM is completely determined by a wave function, or an element of the phase space $\mathbb{H}$ in any other representation. The abstract form of the state, introduced by Dirac [33], is the **ket** $|\Psi\rangle$. The two are related by

$$\Psi(q_1, \ldots, q_N) = \langle q_1, \ldots, q_N | \Psi \rangle. \tag{A.5}$$

The multiplication operator $\boldsymbol{q}$

$$\boldsymbol{q} : \mathbb{H} \to \mathbb{H} : \Psi \mapsto \Phi = \boldsymbol{q}\Psi, \tag{A.6}$$

where

$$\Phi = \boldsymbol{q}\Psi : \mathbb{F} \to \mathbb{C} : q \mapsto \Phi(q) = q\Psi(q), \tag{A.7}$$

is the observable associated with the coordinate q in the Schrödinger representation, which is the spectral representation of the observable as a Hermitian (technically self-adjoint) operator.

In the same representation the canonically conjugate coordinate is represented by the Hermitian operator that is the derivative operator $\boldsymbol{p}$

$$\boldsymbol{p} : \mathbb{H} \to \mathbb{H} : \Psi \mapsto \Phi = \boldsymbol{p}\Psi, \tag{A.8}$$

where

$$\Phi = \boldsymbol{p}\Psi : \mathbb{F} \to \mathbb{C} : q \mapsto \Phi(q) = -i\hbar \frac{\mathrm{d}\Psi(q)}{\mathrm{d}q}. \tag{A.9}$$

The fact that the two coordinates are canonically conjugate is expressed by the commutation relation between the observables as operators

$$[\boldsymbol{q}, \boldsymbol{p}] = i\hbar \boldsymbol{I} \tag{A.10}$$

with $\boldsymbol{I}$ the identity operator on $\mathbb{H}$. A theorem by von Neumann establishes that the representation of conjugate coordinate by multiplication and derivative is a unique representation of a pair of operators satisfying the above commutation relation [61]. It is possible to represent the state of the system as a wave function on the spectrum of $\boldsymbol{p}$, which is called the momentum representation. In that representation the coordinate $\boldsymbol{q}$ is then represented as the derivative operator.

The two observables $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, or $\boldsymbol{p}$, that make up a degree of freedom are inextricably intertwined in QM. This is the most significant difference with CM, where both coordinate and canonical momentum are separate variables that each take on a single numeric real value for any state of a classical system at all times during any dynamical process. The degree of freedom as part of the description of the state of any quantum system is represented by the wave function on the spectrum of one of the two observables as Hermitian operators. If the wave function is given as a function on the spectrum of the coordinate $\boldsymbol{q}$, then all information about the wave function as a function on the spectrum of the conjugate momentum $\boldsymbol{p}$ is already available and can be obtained by the spectral transform of the operator $\boldsymbol{p}$, which is the Fourier transform.

There is no way to split the information in the wave function into a part that has only the information about the coordinate $\boldsymbol{q}$ and another part that has only the information about the conjugate momentum $\boldsymbol{p}$.

It is useful to look at the Hilbert space of QM as a classical phase space with the real and imaginary parts of the wave function as conjugate variables and the complex structure of the Hilbert space as the symplectic structure [49]. However, there is no simple relation between the real or imaginary part by itself of the wave function in the position representation and any observable of a particle that could give it physical meaning.

The **Wigner distribution function** $W_\psi(q, p)$ computed from a wave function ψ [8, p. 29] provides a way to visualize the position and momentum content of a wave function in one construct. The very fact that the Wigner distribution provides two views of the same wave function, smoothly connecting the wave function for position with its Fourier transform for momentum, illustrates the point that the position and momentum content are inextricably connected in the ket.

A.1 Assumptions

We formulate the basic assumptions of QM:

Assumption A.1 *QM is a mechanics of kets, or wave functions, that represent physical systems in terms of their degrees of freedom. The kets are the basic elements of the mathematical formalism of QM.*

The ket can be interpreted as the state of the physical system, but this requires an ontology and an interpretation to provide context, as is discussed in Appendix C.

Assumption A.2 *In QM, the evolution in time of the ket is given by the solution of the first-order differential equation for kets $|\Psi,t\rangle$ or wave functions $\Psi(q_1,\ldots,q_N,t)$ of the system*

$$i\hbar\frac{\partial}{\partial t}\Psi(q_1,\ldots,q_N,t) = \boldsymbol{H}\Psi(q_1,\ldots,q_N,t), \tag{A.11}$$

called the Schrödinger equation (SE). All observable phenomena in physics can be derived from the evolution of kets of physical systems according to the SE.

The construction of the Hamiltonian operator $\boldsymbol{H}$ is an important step in the formulation of the quantum description of any system. The classical Hamiltonian function $H(q,p)$ often serves as a good guide.

Because the SE is a linear equation, its solution is explicitly known in general and it is given in terms of the one-parameter family of unitary operators

$$\boldsymbol{U}(t) = \exp\left(-\frac{it}{\hbar}\boldsymbol{H}\right) = \int_{E_0}^{\infty}\exp\left(-\frac{it}{\hbar}E\right)\boldsymbol{M}(\mathrm{d}E), \tag{A.12}$$

where $\boldsymbol{M}$ is a projection-operator-valued measure on the spectrum $[E_0,\infty[$ of $\boldsymbol{H}$ [66]. This family of operators forms an Abelian group: $\boldsymbol{U}(t_1+t_2) = \boldsymbol{U}(t_1)\boldsymbol{U}(t_2)$. The operators are defined with the spectral representation of the Hamiltonian operator $\boldsymbol{H}$, which is the unitarily equivalent form of the operator where it acts as a multiplicative operator [61,66].

The relativistic description of a quantum system requires the extension of the one-parameter group of time translations Eq. (A.12) to a unitary representation $\boldsymbol{U}(\mathbf{r},t,\Lambda)$ in $\mathbb{H}$ of the 10-dimensional Poincaré Lie group with $(\mathbf{r},t)$ a translation in spacetime and Λ an element of the 6-dimensional Lorentz Lie subgroup of rotations and Lorentz boosts [84,92,45,83]. The transformation of an operator $\boldsymbol{A}$ under the full Poincaré group

$$\boldsymbol{A}_{\mathbf{r},t} = \boldsymbol{U}(\mathbf{r},t,I)^{\dagger}\boldsymbol{A}\boldsymbol{U}(\mathbf{r},t,I) \tag{A.13}$$

is a natural relativistic generalization of the **Heisenberg representation** [33, p. 112].

An important phenomenological element of QM is **Born's rule** [18,19] which provides a connection between the formal description of a quantum system given by the ket and numbers obtained in experiments on the quantum system supposedly described by that ket. The rule is as follows:

> For a quantum system in a pure state described by a ket $|\Phi\rangle$ the measurement of an observable represented by a self-adjoint operator $\boldsymbol{A}$ results in values a being obtained with a probability $p(a)$ that is given by the modulus squared
>
> $$p(a) = |\Phi(a)|^2 \tag{A.14}$$
>
> of the wave function $\Phi(a)$ representing the ket $|\Phi\rangle$ on the spectrum of $\boldsymbol{A}$.

The Born rule is variably used as a definition of the interpretation or the meaning of the wave function, and of the ket, or as an assumption in the formulation of QM to describe an alternative way the ket can change under the effect of interacting with other systems in addition to evolution under the SE as in Assumption A.2. Everybody uses the Born rule, but there is no universal consensus on what its true role is [50,23, 60,91]. In the work of ABN [3,4] and in this paper, the Born rule is derived from from the LvNE Eq. (B.10) and the SE Eq. (A.11), respectively.

A.2 Composite systems

There are important differences between CM and QM in the way the theories handle composite systems. We exhibit the procedure in two theorems; the statement in CM is usually not discussed explicitly because it is considered self-evident.

Theorem A.1 *Consider two systems A and B in CM with states*

$$(q^A,p^A) = (q_1^A,\ldots,q_M^A,p_1^A,\ldots,p_M^A) \tag{A.15}$$

and

$$(q^B,p^B) = (q_1^B,\ldots,q_N^B,p_1^B,\ldots,p_N^B). \tag{A.16}$$

The state of the composite system $C = A+B$ is then given by the vector (q^A,q^B,p^A,p^B) in the direct sum $\mathbb{R}^{2M}\oplus\mathbb{R}^{2N}$ of the phase spaces $\mathbb{R}^{2M}$ and $\mathbb{R}^{2N}$. This state is unique.

Conversely, any state $(q_1^C,\ldots,q_K^C,p_1^C,\ldots,p_K^C)$ of a system C can be decomposed into a state (q^A,p^A) for subsystem A and a state (q^B,p^B) for subsystem B with $K = M+N$. This decomposition is unique.

The theorem is a direct consequence of the structure of classical phase space. As a consequence, every system can be uniquely characterized as a composition of its degrees of freedom as shown in Eq. (A.3).

The situation in QM is very different. The analysis is due to von Neumann [61, §VI.2]. We formulate his result in close parallel with the result in CM.

Theorem A.2 *Consider two systems A and B in QM with respective kets $|\Psi^A\rangle$ and $|\Psi^B\rangle$. The state of the composite system C is given by $|\Psi^C\rangle = |\Psi^A\rangle\otimes|\Psi^B\rangle$ in the tensor product $\mathbb{H}^A\otimes\mathbb{H}^B$ of the phase spaces $\mathbb{H}^A$ and $\mathbb{H}^B$. This ket is unique.*

Conversely (von Neumann's result), given a ket $|\Psi^C\rangle$ of a system C, with associated statistical operator $\boldsymbol{D}^C = |\Psi^C\rangle\langle\Psi^C|$, it is possible to find two statistical operators $\boldsymbol{D}^A$ and $\boldsymbol{D}^B$ for subsystems A and B such that

$$\begin{aligned}
\boldsymbol{D}^A &= \sum_{n=0}^{\infty}\lambda_n|\Xi^A,n\rangle\langle\Xi^A,n| \\
\boldsymbol{D}^B &= \sum_{n=0}^{\infty}\lambda_n|\Upsilon^B,n\rangle\langle\Upsilon^B,n| \\
\boldsymbol{D}^C &= \sum_{n=0}^{\infty}\lambda_n|\Xi^A,n\rangle|\Upsilon^B,n\rangle\otimes\langle\Xi^A,n|\langle\Upsilon^B,n|
\end{aligned} \tag{A.17}$$

This decomposition is unique.

For a general ket $|\Psi^C\rangle$ more than one weight λ_n is different from zero, so that the separation of a system $C = A+B$ does not lead to a unique ket for each of the two subsystems A and B; that only happens for product kets $|\Psi^C\rangle = |\Psi^A\rangle\otimes|\Psi^B\rangle$.

Given two statistical operators $\boldsymbol{D}^A$ and $\boldsymbol{D}^B$, the ket $|\Psi^C\rangle$ can be reconstructed from the sequences of kets $(|\Xi^A,n\rangle)_n$ and $(|\Upsilon^B,n\rangle)_n$ together with the eigenvalues $(\lambda_n)_n$ and a sequence of phases $(\varphi_n)_n$

$$|\Psi^C\rangle = \sum_{n=0}^{\infty}\sqrt{\lambda_n}e^{i\varphi_n}|\Xi^A,n\rangle\otimes|\Upsilon^B,n\rangle. \tag{A.18}$$

The reconstruction of a general state $|\Psi^C\rangle$ from two statistical operators $\boldsymbol{D}^A$ and $\boldsymbol{D}^B$ cannot be completed without the phases $(\varphi_n)_n$.

The proof is well-known and can be found in many textbooks, in addition to von Neumann's book, see for example [51, §11-8]. The kets $(|\Xi^A, n\rangle)_n$ and $(|\Upsilon^B, n\rangle)_n$ are obtained as the eigenvectors of the density operators, which are by definition real and positive, so that the eigenvectors are naturally chosen as real. The decomposition is a special case of the Schmidt decomposition [70, 71]. That decomposition is written without the phase factors because the functions $(|\Xi^A, n\rangle)_n$ and $(|\Upsilon^B, n\rangle)_n$ do not have to be real in general.

These two theorems show that decomposition into subsystems is unambiguous in CM, but it is in general not possible in QM.

A.3 Superposition and entanglement

The phase space and the dynamical equation in QM are both linear: If $|\Psi\rangle$ and $|\Phi\rangle$ are valid kets for a physical system, then $a|\Psi\rangle + b|\Phi\rangle$ is also a valid ket for any two complex numbers a and b; if $|\Psi, t\rangle$ and $|\Phi, t\rangle$ are valid evolutions for a physical system as solutions of the SE, then $a|\Psi, t\rangle + b|\Phi, t\rangle$ is also a valid evolution and a solution of the SE for any two complex numbers a and b. Such a linear combination of states, kets, and wave functions is called a **superposition**.

Superpositions in spaces that are tensor products lead to a new feature that is very different from classical superpositions of waves in water or in the electromagnetic field. Consider a superposition $a_n|\alpha, n\rangle + b_n|\beta, n\rangle$ for a single degree of freedom labeled n. A valid ket for the system with N degrees of freedom $n = 1, \ldots, N$ is the tensor product ket

$$\bigotimes_{n=1}^{N}(c_{\alpha n}|\alpha, n\rangle + c_{\beta n}|\beta, n\rangle). \tag{A.19}$$

This ket is a superposition of the product kets

$$|p(1), 1\rangle \otimes \ldots \otimes |p(N), N\rangle, \tag{A.20}$$

where p is a map $n \mapsto p(n) \in \{\alpha, \beta\}$. There are 2^N such kets. The superposition is a very special one in that it is itself still a tensor product of kets for each degree of freedom or each subsystem. As shown by von Neumann in Theorem A.2, this superposition is very much like a classical superposition of waves and fields.

But superpositions in QM are in no way restricted to such superposition; as a matter of fact, such superpositions are rare because the superposition coefficients must have the special product form $c_{p(1)1} \ldots c_{p(N)N}$. Because of interactions between degrees of freedom or subsystems, the SE quickly evolves tensor product states into states that are no longer tensor products. The kets and wave functions can still be written as superpositions of product kets like the ones in Eq. (A.20), but the expansion coefficients no longer have the product form. Such superpositions can be called **entangled states**, because there is no way to decompose the state of the system into states for the subsystems. The decomposition leads to von Neumann statistical operators for the description of the subsystem as given by Theorem A.2.

In a very interesting paper, Harshman and Ranade [46] illustrate the pervasive and relative nature of entanglement, by showing that any pure state can be written as an entangled state by constructing a set of tailored observables that give the Hilbert space in which the state is defined a tensor product structure in such a way that the pure state is not a product state. Their proof is valid for finite-dimensional Hilbert spaces because they rely on a matrix representation of the observables.

B Non-equilibrium statistical mechanics

To set notations and clarify conventions, we give a short summary of statistical mechanics, both classical and quantum. We add the designation "non-equilibrium" to make clear that we are interested in the dynamics of systems, not in the equilibrium states introduced by Boltzmann and Gibbs [41] in what is now also known as statistical thermodynamics [77].

B.1 Classical mechanics

Consider a classical system with N degrees of freedom. The classical **configuration space** is the space $\mathbb{R}^N$ of all vectors $q = (q_1, \ldots, q_N)$ and the classical **phase space** is $\mathbb{H} = \mathbb{R}^{2N}$ with each point $(q, p) = (q_1, \ldots, q_N, p_1, \ldots, p_N)$ corresponding to a **state** of the classical system. Note that, in the state, the coordinate q_i and its canonically conjugate momentum p_i are specified by independent values. A single degree of freedom in mechanics is described by a coordinate q_i and its velocity $\dot{q}_i$ or canonically conjugate momentum p_i. It then makes sense to write the phase space of a system with N degrees of freedom as shown in Eq. (A.3).

Definition B.1 The **statistical state** of a classical system is a probability measure σ on phase space $\mathbb{H} = \mathbb{R}^{2N}$ with associated probability distribution function (PDF) $\rho(q, p)$ assigning a probability

$$\sigma(S) = \int_S \rho(q, p)\mathrm{d}^N q\mathrm{d}^N p \tag{B.1}$$

to any Borel[22] set $S \subseteq \mathbb{H}$.

A probability measure is normalized so that the probability of the whole space is equal to 1

$$\sigma(\mathbb{H}) = \int_{\mathbb{H}} \rho(q, p)\mathrm{d}^N q\mathrm{d}^N p = 1. \tag{B.2}$$

The **Hamilton flow** of classical dynamics in phase space

$$(q_t, p_t) = F_t(q_0, p_0), \tag{B.3}$$

i.e. the solution of Hamilton's equations, then defines the evolution of the statistical state as [43, 53, 97, 57]

$$\rho_t(q, p) = \rho\big(F_{-t}(q, p)\big). \tag{B.4}$$

The evolution σ_t of the measure follows from inserting Eq. (B.4) into Eq. (B.1).

It follows from the above definition of the time evolution that the density ρ_t of the statistical state σ_t satisfies the differential equation

$$\frac{\partial \rho_t(q, p)}{\partial t} = \{H(q, p), \rho_t(q, p)\}, \tag{B.5}$$

called the Liouville equation (LE) [97]. Here

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} \tag{B.6}$$

is the Poisson bracket.

The evolution of the statistical state under the LE Eq, (B.5) is completely equivalent to the states in phase space

[22] On spaces like $\mathbb{R}^{2N}$ Borel sets are sets that can be created as unions and intersections of open hypercubes.

moving under the Hamiltonian flow Eq. (B.3) or to evolution under Hamilton's equations.

The interpretation of the statistical state in the literature varies: Boltzmann [41, §3.2] conceives of a classical system, for example a gas, as a single system of a large number of particles, with the statistical state characterized by specifying that the state of the gas is confined to a given region, a subset $S \subset \mathbb{H}$ of phase space. This makes the PDF the index function ι_S of that set, equal to one for points in the set and zero otherwise.

Gibbs [41, §3.3], on the other hand, considers an ensemble of identical systems with coordinates chosen randomly with probability $\rho(q,p)\mathrm{d}^N q\mathrm{d}^N p$. This probability is interpreted as frequency over the ensemble or as an average over time [41, §3.3.4]. While the latter view is relevant for equilibrium states, it is not meaningful in the context of dynamically evolving non-equilibrium states.

As an example of a statistical state, the Boltzmann partition for a gas in a container with volume V can be defined as a finite measure by constructing the sets $S(E)$ in phase space with total energy E. For an ideal gas, this energy is the sum of the kinetic energy of all atoms or molecules in the gas. Then the PDF is given by weighing the sets with the Boltzmann factor at temperature T

$$\rho_{\mathrm{Boltz}}(q,p) = C\exp\left(-\frac{E}{kT}\right)\iota_{S(E)}(q,p) \tag{B.7}$$

with C the normalization constant to satisfy Eq. (B.2). Physically more accurate models of the gas add interactions between the atoms or molecules and between the atoms or molecules and the walls of the container, which results in modified sets $S(E) \subseteq \mathbb{H}$.

B.2 Collective variables from the statistical state

We propose to view the probability measure σ Eq. (B.1) in a third way that is a blending of the two views of Boltzmann and Gibbs summarized in Section B.1. Consider the fact that the measurement of a macroscopic observable, like pressure, of a macroscopic system, like a gas, is never accomplished by explicitly and meticulously recording the position and momentum of every atom or molecule in the system and then computing the macroscopic variable. Rather, experiments measure macroscopic dynamical variables with macroscopic devices that contain macroscopic controls such as size, shape, position, pressure, volume, and temperature, and these controls then are associated with the value of observables of the system under observation.

While the atomic hypothesis is of great value in the theoretical description of observed phenomena and in deriving governing laws between observations, it is not directly relevant as part of the practice of making observations. The frequency interpretation of the statistical state as a probability measure σ is therefore somewhat artificial: The measure σ assigns a probability $\rho(q,p)\mathrm{d}^N q\mathrm{d}^N p$ for each microscopic state (q,p), but in practice one never obtains these states as samples from a distribution as specified in the frequency interpretation. Therefore, it is possible to think of the probability measure σ itself as the mathematical description of the macroscopic state, i.e. we take the meaning of Definition B.1 to be:

1. In non-equilibrium statistical CM, the mathematical description of the state of a macroscopic physical system with N degrees of freedom is the statistical state of the system, i.e. the probability measure σ on the system's phase space $\mathbb{H} = \mathbb{R}^{2N}$.
2. The macroscopic observables (volume, pressure, temperature) are encoded in the probability measure σ and their evolution follows from that of σ_t as derived by Eq. (B.4) from the underlying flow $F_t(p,q)$ in $\mathbb{H} = \mathbb{R}^{2N}$, Eq. (B.3).

B.3 Quantum mechanics

The statistical state in quantum mechanics was introduced by von Neumann [61] to describe a state for a quantum system that is incompletely specified, i.e. as a mixture or Gemenge [23, p. 21]. It is defined as the positive symmetric operator

$$\boldsymbol{D} = \sum_{n=0}^{\infty} p_n \boldsymbol{P}_n = \sum_{n=0}^{\infty} p_n |n\rangle\langle n| \tag{B.8}$$

specifying the system to be in one of a number of kets $|n\rangle$ with positive weights p_n. The normalization that the total probability equal 1 then requires the operator to have trace equal to 1

$$\mathrm{Tr}\boldsymbol{D} = \sum_i p_i = 1. \tag{B.9}$$

The operator $\boldsymbol{D}$ is called the **statistical operator**, **density operator**, or **density matrix**.

The dynamical evolution of the statistical state is given by the Liouville-von-Neumann equation (LvNE)

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D}_t = [\boldsymbol{H}, \boldsymbol{D}_t]. \tag{B.10}$$

If the initial statistical operator is written with an orthonormal set of kets $(|n\rangle)_n$, then its evolution under the LvNE

$$\boldsymbol{D}_t = \sum_{n=0}^{\infty} p_n |n,t\rangle\langle n,t| \tag{B.11}$$

is equivalent to the evolution of the kets according to the SE and the numbers p_n do not change in time, as in the classical case.

As a positive symmetric operator, the statistical operator can always be diagonalized, resulting in a unique composition of the statistical operator as a mixture of orthogonal states with positive weights p_n. But, as pointed first out by Schrödinger [75,76,63,23], it is not possible to associate a unique set of pure states that are linearly independent, but not necessarily orthogonal, with a given statistical operator. Therefore, it is not possible to directly interpret the numbers p_n as probabilities with the meaning of frequency of occurrence in experiments on ensemble of systems in the statistical state described by $\boldsymbol{D}$. Hence there is a clear distinction between the evolution of the statistical operator governed by the LvNE and pure states governed by the SE. This situation is different from the situation in statistical CM and that is the core message of the statistical interpretation of QM [7,3,4]: It is not possible to clearly and uniquely distinguish statistical probabilities in mixtures from the innate probabilistic nature of pure states; there is only one single unavoidable probability notion in QM and it is not a purely statistical probability as it is in CM.

C Empiricism and realism

We summarize the context of concepts relevant to the measurement problem in QM, see, for example, the book by de Muynck [60]. The analysis by ABN [3, 4] shows that the Born probability rule can be derived from the dynamics of QM provided the complexity necessarily associated with measurement instruments is given proper consideration. This approach is not new, for example d'Espagnat [27, §16.2] explores the options of a single principle, namely the SE, in QM for changing the state, represented by a ket or a wave function. The relative state formulation of Everett [39] is also based entirely on the SE as the only principle for changing states. The work by ABN appears to be the most exhaustive analysis to date that shows that evolution in QM under the LvNE leads to a unique outcome, which contradicts Everett's claim [39, p. 457] that "It seems that nothing can ever be settled by such a measurement."

C.1 Newton-Maxwell ontology

In Newtonian mechanics the ontology is clear: The world consists of "objects with substance" that are described by coordinates and their velocities subject to Newton's law of force. The mass of the object gives a measure of the amount of substance. Newtonian mechanics is scale invariant in that the objects can be described at multiple levels, depending on desired accuracy, either as monolithic objects with substance, for example planets, or as composed of a number of moving parts with substance, such as cars, or, applying the atomic hypothesis, as assemblies of atoms with substance. The essence of the ontology is that the degrees of freedom are described by pairs of variables that at all times take on values that are real numbers, one degree of freedom being specified by two values. By using the ontology as a framework for thinking about the world and the physics processes taking place in it, the physicist can accurately and efficiently apply the formalism of Newtonian mechanics for computing descriptions of observed processes and make predictions of planned or expected future observations.

The same ontology exists for Maxwell's theory of electromagnetism, but now applied to the ontological element of a "field of force" defined in all of space. The field is described by a dynamical coordinate and its associated velocity, both taking on a numeric value, defined in every point of space. Einstein's theories of relativity are consistent with this ontology, both the special theory for particles and fields, and the general theory adding space to the list of ontological elements.

We refer to this ontology as the **Newton-Maxwell ontology**.

When Einstein does the statistical analysis [35] of light interacting with matter, he concludes that the observed behavior is consistent with that of a gas of photons. In Bohr's development of a description for atomic spectra [14], a theory of the Periodic Table of Elements [15], and an explanation of the chemical bond [16], he uses electrons moving in classical circular and elliptic orbitals selected by the "quantum condition". It is clear that Einstein and Bohr both work with the Newton-Maxwell ontology.

Einstein [36] summarizes his view by stating that there are two options:

> (a) The (free) particle really has a definite position and a definite momentum, even if they cannot both be ascertained by measurement in the same individual case. According to this point of view, the ψ-function represents an incomplete description of the real state of affairs.

and

> (b) In reality the particle has neither a definite momentum nor a definite position; the description by ψ-function is in principle a complete description. The sharply-defined position of the particle, obtained by measuring the position, cannot be interpreted as the position of the particle prior to the measurement. The sharp localisation which appears as a result of the measurement is brought about only as a result of the unavoidable (but not unimportant) operation of measurement.

Both Einstein [35] and Bohr [14] agreed on the appropriateness of the notion that physical systems have properties that are described mathematically by **numbers**, individual values that are captured during the measurement process [50, 42], in particular for the pair of canonically conjugate variables making up a degree of freedom. They were arguing about whether the systems, like particles, have these values before and after measurement, or only during the measurement process. This view was prevalent and very successful at the time and is beautifully described in the works by Sommerfeld [82] and Born [20]. While Bohr advocated a conceptual complementarity [17] to understand the principles of QM and the observations of quantum phenomana, Einstein wanted something more precise [37].

The ideas introduced by Einstein and Bohr remain valid, even though the precise calculations to quantify the photoelectric effect, the lines in atomic spectra, and the structure of molecules are now done with the formalism developed in 1926 instead of the application of the quantum condition. It is natural that Einstein and Bohr, and others, considered the 1926 formalism as a new and improved, and more complicated and highly unfamiliar, version of the quantum condition. They assumed that the prevailing Newton-Maxwell ontology was still applicable [17]. We argue in this paper that it is not applicable.

C.2 State-measurement connection

In CM it is an experiential fact that measurements exist that provide direct observation of the values taken by coordinates and velocities, at least for measurements on sufficiently simple systems. This leads to the positivist idea that physics theories can be and should be formulated such that experimental observations are the foundation for all concepts [29]. The historically most important example is the observation of the planetary positions (and velocities from multiple positions) by observing the Sun's light reflected by the planets. The second prominent example is the definition of the electric and magnetic field strengths as the force felt by an infinitesimal test charge.

Note that this does not preclude that with complex systems, such as automobile engines, some variables may not be observable in a similarly direct way, but their values can be reconstructed, by using the law of force, from the values obtained for other dynamical variables that can be observed.

In their debate, Einstein and Bohr take the observation of localized flashes produced by photons or electrons to be direct information about the photons or electrons [50, 42]. The analysis by ABN [3, 4] conclusively shows that even the simplest experimental observation of a quantum mechanical system is

a complex multi-step process. It is unwarranted to conclude that the state of a macroscopic device in an experiment provides a direct view on the microscopic quantum system that initiated the event.

On the basis of the Newton-Maxwell ontology and with the assumption that measurements in QM provide direct information about the state of quantum systems, Einstein and Bohr argued about how the observed facts could be derived from this ontology. In particular, they argued about whether the values for coordinates and velocities (or momenta) were present all the time or just during the measurement [50,42]. They did not consider the possibility [36] that the ontology does not apply to QM and that all experimental observations in QM may have to be derived from the dynamics by complex computations of quantum systems as shown, for example, by ABN. Born's probability rule [18,19] provides an operational definition of the quantum mechanical state, ket, and wave function, supporting the Newton-Maxwell ontology, but it does not explain the connection.

The theory of measurement presented by ABN is rigorous and minimalist. No special assumptions are introduced, like hidden variables or spontaneous wave function collapse. QM is assumed to be irreducibly probabilistic [3, p. 6] and the analysis focuses on the ideal projection-operator measurement as defined by von Neumann [61]. Contrary to the Copenhagen interpretation, where two distinct processes are assumed, the SE for dynamical evolution and the Born rule for the measurement process, the measurement process is derived in full detail from the dynamical evolution of the statistical operator governed by the LvNE Eq. (B.10). That gives the evolution for both the observed quantum system and the macroscopic measurement apparatus.

We briefly summarize the stages in the dynamical process that are identified in the analysis by ABN. They identify three component systems in the analysis:

1. The microscopic quantum system S to be measured.
2. The macroscopic measurement apparatus and pointer system M that will register, after some time, an outcome correlated with the state of S.
3. A macroscopic system B that provides a heat bath to ensure that the measurement is irreversible as a dynamical process.

The system is described by a statistical operator $\boldsymbol{D}$, given by Eq. (B.8), that evolves under the LvNE Eq. (B.10). The dynamics is governed by the Hamiltonian with an important role played by the terms describing the interactions between the different systems. The different timescales in the process are carefully analyzed and described by ABN with the following summary of the process [3, Table 1, p. 103] [4]:

1. Preparation – create a metastable state in the apparatus or pointer M; the statistical operator is $\boldsymbol{D} = \boldsymbol{D}^S \otimes \boldsymbol{D}^M \otimes \boldsymbol{D}^B$.
2. Initial truncation – decay off-diagonal blocks of statistical operator $\boldsymbol{D}^M$ of the pointer.
3. Irreversible truncation – further interaction between the pointer M and the bath B ensure that there is no recurrence of the off-diagonal blocks of $\boldsymbol{D}^M$.
4. Registration – $S-M$ correlation is built up in diagonal blocks of the statistical operator $\boldsymbol{D}^M$ of the pointer to correlate with the diagonal elements of $\boldsymbol{D}^S$ in a process that is similar to a phase transition [3, §7] and involves a weak coupling to the bath B. This process ensures that the pointer M reflects the state of the system S as it was at the start of the measurement process: $\boldsymbol{D}^S_{aa} \leftrightarrow \boldsymbol{D}^M_{aa}$.
5. Sub-ensemble relaxation – interaction terms in the Hamiltonian of the pointer ensure a consistent evolution in ensembles, so that sub-ensembles can be identified that ultimately lead to the outcome of a single run.
6. Reduction – gain of information in the pointer M about the initial state of S with probabilities in accordance with Born's rule.

The majority of these steps will appear plausible to anyone with experience in studying the measurement problem, except possibly item 5. The need for the analysis of ensembles and sub-ensembles comes from Schrödinger's observation [75, 76,63,23] the statistical operator cannot be uniquely decomposed in terms of pure states, as discussed in Section B.3. The reader is referred to ABN [4] for an in-depth and lucid discussion.

C.3 Heisenberg-Dirac ontology

> Bohr expected the answer would follow from a direct analysis of the definition of the idealized concepts, Heisenberg argued that the answer was hidden in the formal structure of the theory and that a closer scrutiny of this structure would bring it to light.
>
> Commentary of Rosenfeld in 1971 reproduced from [93, p. 58]

Given Newton's objects with substance governed by Newton's law of force as the ontology of mechanics and the electromagnetic fields of force governed by Maxwell's equations as the ontology for electromagnetism, it stands to reason to consider, similarly, the object governed by the Schrödinger equation as the basic element in the ontology for QM: That means taking the abstract ket introduced by Dirac [33], or the wave function in any representation, to be of the same quality of being as the objects with substance in Newtonian mechanics and the fields of force in Maxwell theory. We refer to this ontology as the **Heisenberg-Dirac ontology**.

d'Espagnat [29] discusses ontology of QM in depth, but his analysis remains within the Newton-Maxwell ontology. He does not follow Heisenberg, who, when he was thinking about tracks in the bubble chamber [50,42], questions the complexity of the measurement process and hoped to find understanding in the mathematical formalism as opposed to in experimental observations, as indicated by Rosenfeld's remark quoted above.

What if everything is made of kets? What if there are no particles, no fields, no systems that can be described by degrees of freedom that consist of coordinate-momentum pairs that take on values that are real numbers?

This brings us to a point where we are led to accept as fundamental, realist postulates of QM the two assumptions in Appendix A.1 at face value:

1. The **ontological building block of QM is the ket**.
2. The ket is governed at all times by the SE.

A ket is not a number. Since experiments obtain numbers, it follows that measurements in QM cannot measure the ket of any quantum system; some assembly is required.

D Finite measures on Hilbert spaces

The study of measures on infinite-dimensional spaces is the subject of stochastic analysis, a field in mathematics that was initiated by the work on the space of paths of Brownian motion by Norbert Wiener in 1923 [94].

The first counterintuitive property one encounters with measure on and volume in infinite-dimensional spaces is the fact that there is no translation-invariant measure like the Lebesgue measure on an infinite-dimensional space [52, p. 32]. For the mathematically inclined reader, the fundamental reason is that the unit ball in an infinite dimensional space is not compact.

Theorem D.1 *There does not exist a meaningful translation invariant measure on an infinite-dimensional Hilbert space.*

Proof Consider [24, p. 5] an orthonormal basis $(|\Phi_n\rangle)_n$ in the Hilbert space and a ball B_n with radius 1/2 centered at $|\Phi_n\rangle$. Assume that a translation invariant measure λ exists. Then the measure of all these balls is the same $V = \lambda(B_n)$ by translation invariance, because they are all just translations of the first one B_0. It is easy to see that all these balls fit into a bigger ball B with radius 2 centered at the origin and that none of them overlap.

A proper measure must have the property that the volume of a set B is larger than the sum of the measures of any collection of non-overlapping sets B_n contained in B. Thus we must have

$$\sum_{n=0}^{\infty} \lambda(B_n) \leq \lambda(B). \tag{D.1}$$

It follows that $\lambda(B)$ is infinite as the sum of an infinite number of equal terms.

This contradicts another requirement of reasonable measures, namely that the measure of a bounded set be finite. Thus the only measure satisfying all requirements is one that assigns the measure zero to all balls. It follows easily that that measure must be identically zero. □

Note that the proof uses translation, but the problem really lies with rotations: It is possible to keep turning to new directions in an infinite dimensional Hilbert space putting new balls B_{N+1} that do not overlap with any ball $B_0, \ldots, B_N$ we already have, while still remaining inside the big ball B.

Even though there is no σ-finite, translation invariant measure, it is possible to define finite measures on infinite dimensional spaces, like a Hilbert space, with all the right properties [80,11]. A finite measure is one that assigns to the total space a finite volume, which is chosen to be 1 by convention. A probability measure is an example of a finite measure.

The practical reader may suggest that there is no need to work with infinite-dimensional spaces, as any computations can be carried out in spaces with a finite number of dimensions to get approximations as close as needed. However, the issue is not confined to infinite-dimensional spaces: It does show up in spaces with a large number of dimensions and requires special attention to implement efficient and accurate algorithms [87,58]. The methods developed to handle the infinite-dimensional case provide ways to build such efficient algorithms for the case with a very large number of dimensions.

In the literature on measure theory, with λ denoting the Lebesgue measure, the Lebesgue integral of a function f over a set $S \in \mathbb{R}^n$ is written in various ways as [67,81]

$$\int_S f \mathrm{d}\lambda = \int_S f(x) \mathrm{d}\lambda(x) = \int_S f(x) \lambda(\mathrm{d}x) = \int_S f(x) \mathrm{d}^n x, \tag{D.2}$$

where the last form is identical to the way Riemann integrals are written. For general measures, any of these notations, except the last one, can be found in the literature.

The best-known measures on infinite-dimensional spaces are Gaussian measures γ [80,11]. These are defined by specifying the mean Θ as an element of $\mathbb{H}$

$$|\Theta\rangle = \int_{\mathbb{H}} |\Psi\rangle \gamma(\mathrm{d}|\Psi\rangle) \tag{D.3}$$

and the covariance Γ as an operator on $\mathbb{H}$

$$\langle \Xi|\Gamma|\Upsilon\rangle = \int_{\mathbb{H}} (\langle\Psi| - \langle\Xi|)(|\Psi\rangle - |\Upsilon\rangle)\gamma(\mathrm{d}|\Psi\rangle). \tag{D.4}$$

The covariance operator must be symmetric, positive definite, and trace class [80,11], i.e.

$$\Gamma > 0 \qquad \mathrm{Tr}\Gamma < \infty \tag{D.5}$$

for the Gaussian measure to exist and have support in $\mathbb{H}$ and not some larger space.

In finite dimensions, sets of measures zero can be safely ignored in physics. For example, the Lebesgue measure of individual points and curves in a two-dimensional space is zero: $\lambda(\{(x_1, x_2)\}) = 0$ and $\lambda(\{(x_1(t), x_2(t)) | t \in [0,1]\}) = 0$, respectively. A remarkable property of measures on infinite dimensional spaces is that sets of measure zero can be big.

For example, the space of all kets $|\Psi\rangle$ in Hilbert space with finite expectation value $\langle\Psi|\Gamma^{-1}|\Psi\rangle < \infty$ of the covariance operator Γ is called the **Cameron-Martin space** $\mathbb{E}_\gamma$ of the Gaussian measure γ. It has measure zero, $\gamma(\mathbb{E}_\gamma) = 0$ [11, thm 2.4.7]. To illustrate the "size" of the Cameron-Martin space, consider the Wiener Bridge measure on the Hilbert space $L^2([0,T])$ of square-integrable paths from $x = 0$ at time $t = 0$ to $x = 0$ at time $t = T$ [11,52]. This measure is Gaussian and the covariance operator Γ has eigenvalues proportional to $1/n^2$, so that it is trace class. The paths that count, i.e. that provide the weight of the measure, are paths that are Hölder continuous of order 1/2. These are paths that are continuous and almost nowhere differentiable. They are the paths of Brownian motion. They carry the full weight of the measure in that the measure of all such paths is 1. All paths that are in $L^2([0,T])$ and that are not continuous or that are less smooth than Hölder continuous of order 1/2 form a set of measure zero; these rougher paths carry no weight for the Wiener Bridge measure. The subset of paths that are continuous and almost everywhere differentiable turn out to have finite expectation value for the covariance Γ and thus form the Cameron-Martin space of the Wiener Bridge measure, which has measure zero. In other words, these smoother paths (differentiable) do not carry any weight in the Wiener Bridge measure either. That means they do not contribute to any integral.

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