Development of Geometry in the 19th Century

Andrew Kriehn

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17th Century

- Descartes (1596 - 1950) and Fermat (1607 - 1665) create Analytic Geometry
  - Fermat first circulated a manuscript in 1636 (based on results acheived in 1629) which Descartes used in his work in 1637.
- Liebniz (1646 - 1716) and Newton (1642 - 1726) create Calculus around 1684
- Desargues (1591 - 1661) is the first to systematically study Projective Geometry
  - This is pretty much ignored until 1845

18th Century

- Attempts to prove the parallel postulate
  - Saccheri (1667 - 1733)
    - "The hypothesis of the acute angle is absolutely false; because it is repugnant to the nature of straight lines."
  - Lambert (1728 - 1777)
    - "Proofs of the Euclidean postulate can be developed to such an extent that apparently a mere trifle remains. But a careful analysis shows that in this seeming trifle lies the crux of the matter; usually it contains either the proposition that is being proved or a postulate equivalent to it."

18th Century

- Euler (1707 - 1783) develops Analytic Geometry to its modern form
  - Functional notation
  - Heavy use of coordinates
  - Theory of curves in general, rather than just conic sections
  - Graphical study of trigonometric functions
  - Parametric curves
  - First textbook exposition of solid analytic geometry, both algebraic and transcendental
    - Notably, quadric surfaces

18th Century

- Legendre (1752 - 1833):
  - Worked on number theory and Calculus
    - In particular, developed elliptic integrals, beta function, and gamma function
  - Also published *Elements of Geometry*, which was widely adopted as a substitute for Euclid's *Elements*
  - Includes appendices on trigonometry, including spherical trigonometry
  - Saccheri-Legendre theorem: In neutral geometry, the sum of angles in a triangle is less than two right angles.
  - Any geometry satisfying this property is called "Legendrian"

18th Century

- Monge (1746 - 1818):
  - Father of Descriptive Geometry
  - Developed and systemetized solid Analytic Geometry and elementary Differential Geometry
  - Established the École Polytechnique
    - Served as both an administrator and a teacher
  - Wrote textbooks for the reformation of the mathematics curriculum
  - Pupils went on to write many more textbooks in Analytic Geometry

18th Century

- Carnot (1753 - 1823):
  - Also worked to establish the École Polytechnique, though he never taught
  - Alongside Monge as a founder of modern pure Geometry
  - Wrote three important works in Geometry:
    - *De la Correlation des Figures de Géométrie*
    - *Géométrie de Position*
    - *Essai sur la Théorie des Transversals*

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The Golden Age of Mathematics

- Introduction of:
  - Non-Euclidean geometries
  - N-dimensional spaces
  - Noncommutative algebras
  - Infinite processes

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Definition

- First studied during investigations of perspective
- Desargues and Kepler (independently) introduced the idea of a point at infinity.
- Projective Geometry is the study of geometric properties that are invariant with respect to projective transformations
  - I.e., bijections that map straight lines to straight lines

Poncelet

- Lived 1788 - 1867
- Entered the army corps of engineers in time for the 1812 campaign
- Taken prisoner and wrote *Applications d'Analyse et de Géométrie* while in prison
  - Wasn't published until 1862, despite being meant as a predecessor to his next work
- Wrote his more celebrated, synthetic work, *Traité des propriétés projectives des figures* in 1822
- Sought to make Synthetic Geometry as general as possible

Poncelet

- Promoted a "Principle of Continuity"
  - "The metric properties discovered for a primitive figure remain applicable, without other modifications than those of change of sign, to all correlative figures which can be considered to spring from the first."
- Carried this principle to points at infinity (which he called "ideal points")
  - 2 lines now always intersect at one point, either real or ideal
- Carried it further to "imaginary" points
  - Hence a circle and a line always intersect, whether at real or imaginary points
- Proved that 2 circles always meet at a certain 2 imaginary ideal points

Chasles

- Lived 1793 - 1880
- Studied at the École Polytechnique and was appointed as a professor there in 1841
- Rediscovered Desargues in 1845
- Emphasized the role of cross ratios of colinear points in projective geometry
- Later in life, began the study of Enumerative Geometry

Enumerative Geometry

- Branch of Algebraic Geometry that seeks to determine the number of solutions of algebraic problems by geometric interpretation.
  - 2 The number of lines meeting 4 general lines in space
  - 8 The number of circles tangent to 3 general circles (the problem of Apollonius)
  - 27 The number of lines on a smooth cubic surface (Salmon and Cayley)
  - 2875 The number of lines on a general quintic threefold
  - 3264 The number of conics tangent to 5 plane conics in general position (Chasles)
  - 609250 The number of conics on a general quintic threefold
  - 4407296 The number of conics tangent to 8 general quadric surfaces (Fulton)
  - 666841088 The number of quadric surfaces tangent to 9 given quadric surfaces in general position in 3-space (Schubert 1879.

Steiner

- Lived 1796 - 1863
- Heavily disliked analytic methods
- Remarkable for his great generality and rigor
- Showed that all of Euclidean geometry can be done with a straightedge and one circle (instead of a compass)
- Used the principle of point-line duality throughout as a fundamental property of Geometry
- Obtained synthetic solutions to problems which, analytically, would require the Calculus of Variations

von Staudt

- Lived 1798 - 1867
- Saw Projective Geometry as having no place for distances, angles, and perpendiculars
- Instead, used the cross ratio as a fundamental measurement in Projective Geometry
- Also discovered the relation that a conic section establishes between poles and polars
    - This relation is, in some sense, more fundamental than the conic itself, and can thus be used to define the conic

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Plücker

- Lived 1801 - 1868
- Believed algebraic methods were superior to the purely geometric approach of Poncelet and Steiner
- Abbreviated notations, as Lamé did.
  - For example, the intersection of two circles is normally written

$$x^2 + y^2 + ax + by + c = 0$$

$$x^2 + y^2 + dx + ey + f = 0$$

    This becomes $C + \mu D = 0$.
  - Used this to explain the Cramer-Euler paradox
- Was one of four discoverers of homogeneous coordinates
  - Others were Feuerbach, Möbius, and Bobillier

Plücker

- Consideration of $aX + bY + cZ = 0$ to establish, analytically, the duality of points and lines
- Went on to show that every curve can be regarded as having a dual origin

Cayley and Cauchy

- Cayley (1821 - 1895):
  - Began ordinary Analytic Geometry of n-dimensional space (1843)
  - Extended many theorems of 3-space to theorems of 4-space
- Cauchy (1789 - 1857):
  - Published a paper considering analytical points and analytical lines in spaces with more than 3 dimensions

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Lobachevsky

- Lived 1793 - 1856
- First presented about the parallel postulate in 1826, but the paper is lost
- During the next few years, became convinced of the postulate's independence
- 1829: Became the first mathematician to publish a geometry based on the denial of the parallel postulate
- Called this new geometry "imaginary geometry"
- Published 3 more full accounts over the next two decades
  - 1835: *New Foundations of Geometry*
  - 1840: *Geometrical Investigations on the Theory of Parallels*
  - 1855: *Pangeometry*

Bolyai

- Farkas Bolyai was friends with Gauss from a young age
- Both picked up an interest in the parallel postulate
- Gauss decided it was a consistent geometry, but never published anything
- Bolyai proved many theorems in Hyperbolic Geometry, but was always trying to reach a contradiction
- His son, Janos Bolyai (1802 - 1860), picked up the task from him
  - Published in an appendix to Farkas' work
  - Gauss approved, but never gave his support in print
  - Lack of support and Lobachevsky's publication in 1840 left him discouraged, so he never published again.

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Gauss

- Lived 1777 - 1855
- 1827: Initiated the study of Differential Geometry
  - The study of Geometry "in the small"
- Defined what would be called Gaussian curvature
- Used Euler's parametric equations of a surface to discover properties of curves drawn on the surface
- Looked at arc lengths, coordinates, and invariants under transformations

Riemann

- Lived 1826 - 1866
- 1854: Delivered a lecture, "On the Hypotheses which Lie at the Foundation of Geometry"
- Urged a global view of Geometry as a study of manifolds with any number of dimensions in any kind of space
- Should not deal with lines, points, or spaces, but rather sets of n-tuples that follow certain rules
- In particular, how one finds infinitisemal distance
- Used it to give a formula for the Gaussian curvature of a surface

Models

- Riemann showed that a certain Geometry ("Riemannian Geometry") could be modeled on the surface of a sphere
- Beltrami (1835 - 1900) likewise showed that Hyperbolic Geometry could be modeled on the surface of a pseudosphere

Klein

- Lived 1849 - 1925
- Taken in by group theory, created the Erlanger Program to classify geometries via Algebra
- Correlates Geometry as the study of invariants under certain transformations with the groups of those transformations

Algebraic Geometry

- Probably don't have time for this