Traces and Ancient Egypt

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Outline

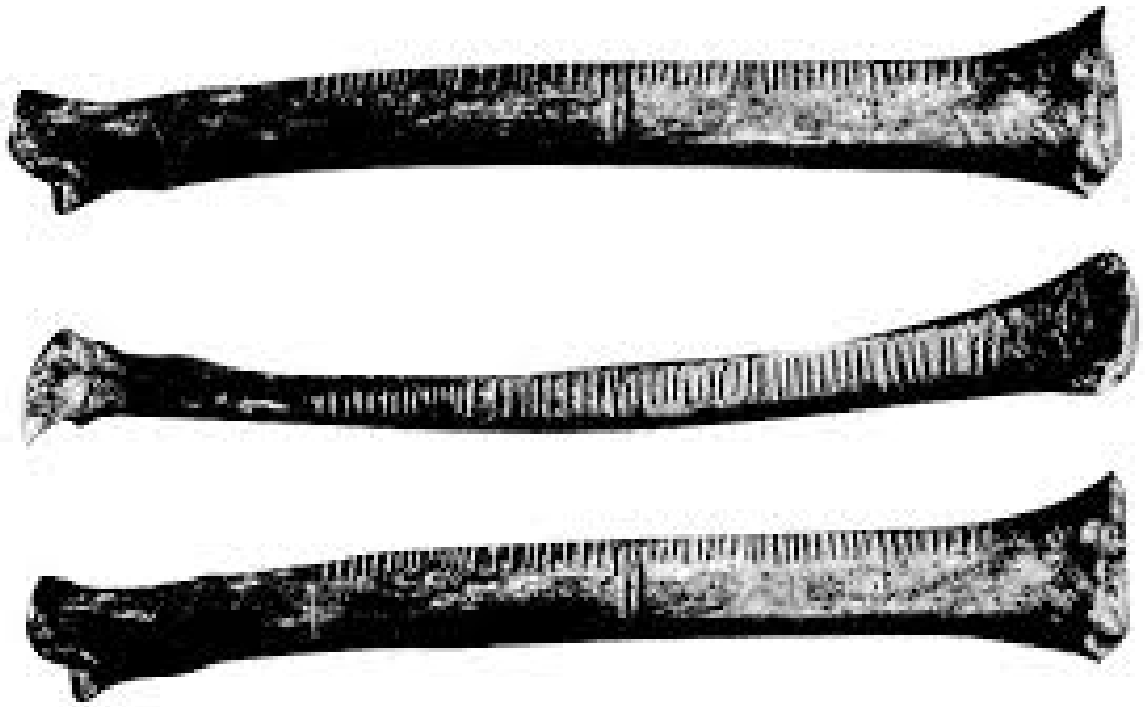
Concepts and Relationships

- Modern mathematics involves forming questions about abstract objects and then establishing its validity – this was not always so
  - For millennia, mathematics was the science of numbers, magnitudes, and forms
- The earliest examples of mathematics come from non-literate cultures some $35,000$ years ago.
  - Because of the pervasion of oral tradition, artifacts say little about *how* they did math, but simply indicated some extent of what they knew

Concepts and Relationships

- Concept of number
  - May have started by obsering sameness of quantities (2 hands, 2 feet, etc.) or differences (one duck vs. many ducks)
  - Took a long time! Many ancient cultures could only count to 2, anything more became 'many'
  - Eventually moved to 'representing' number – likely via sign language. When fingers became insufficient, many cultures used objects
    - Moravia wolf-bone. $30,000$ years old. Has 55 carved notches.
    - Key example of *tallying*

Moravia wolf-bone

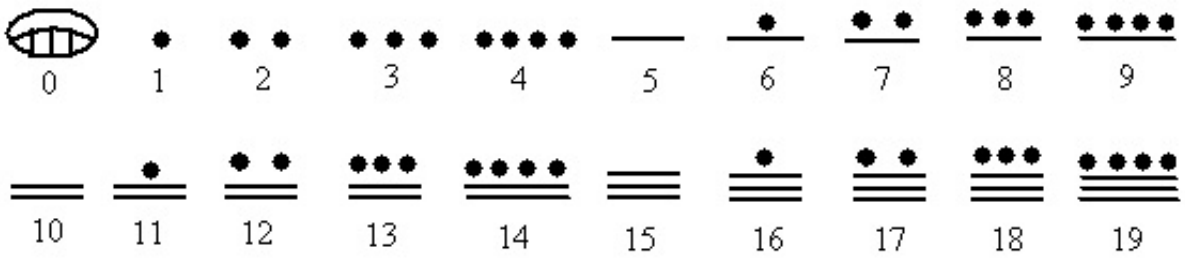


Early Number Bases

- Lots of options:
  - Binary - Base 2
  - Ternary - Base 3
  - Quinary - Base 5
  - Decimal - Base 10
  - Vigesimal - Base 20
  - Sexagesimal - Base 60
- The most common were bases 5 and 10.
- Studies have shown that approximately $\frac{2}{3}$ of American Indian tribes used either base 5, base 10, or a mixture.

Maya of the Yucatan

- One of the first translated items of the Maya was their incredible calendar
- Used primarily a base 20, but had base 5 as an auxiliary base



- Their numbers were expanded in the following form:

$$\# = a_1 + a_2(5) + a_3(20) + a_4(18 \cdot 20) + a_5(20 \cdot 18 \cdot 20) + \ldots$$

- Note - the half-open eye was not reserved for the 'number' 0, but used for a missing position in their numeric expansions

Spacial Relationships

- Ancient pottery, weaving, and basketry suggest knowledge of congruence and symmetry
  - Could be early examples of geometry
  - Without written language, formalization of these concepts seems impossible

Vessel from Mesopotamia (4,500 - 4,000 BCE)

Spacial Relationships

- Why did so many ancient cultures use congruence and symmetry in their constructions?
  - Some historians suggest it was done strictly for the aesthetic beauty – a motivation that still inspires mathematicians today
  - Others suggest that it was strictly for the practical needs
  - Probably a mixture of both, but this serves as a good example of how far back mathematical historians take the Pure vs. Applied debate

Outline

Ancient Egypt

"Sesostris ... made a division of the soil of Egypt among the inhabitants. ... If the river carried away any portion of a man's lot, ... the king sent persons to examine, and determine by measurement the exact extent of the loss. ... From this practice, I think, geometry first came to be known in Egypt, whence it passed to Greece."

– Herodotus, on his 450 BCE visit to Egypt

The Era and the Sources

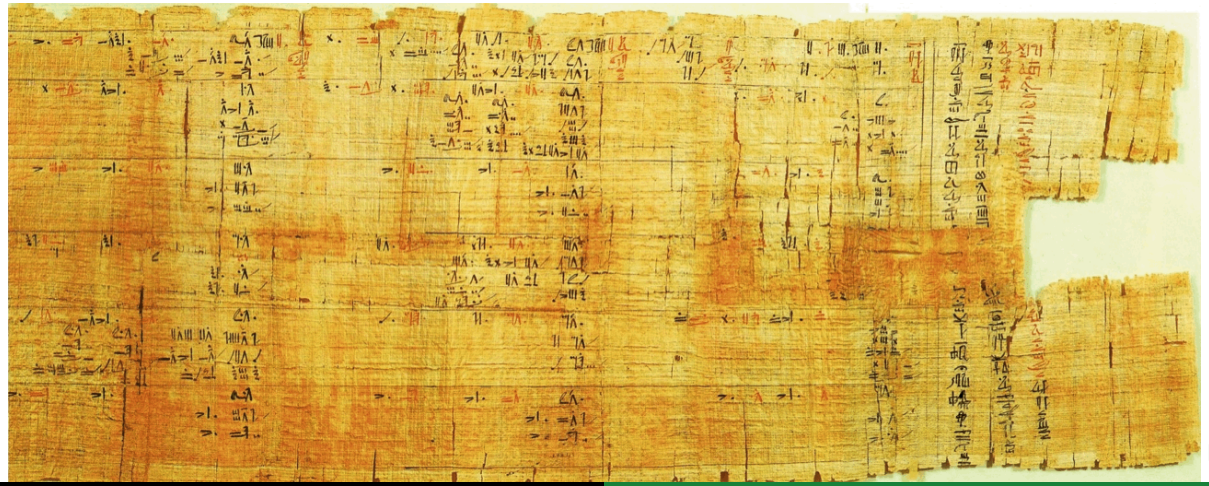
- In 450 BCE, Herodotus of Greece visited Egypt and chronicled his adventure
  - Held that geometry originated in Egypt due to practical needs of surveying the Nile's flooding of the river valley
    - A practical, applied origin
- In 350 BCE, Aristotle visited Egypt as well
  - Also claimed that geometry originated in Egypt and attributed its creation to the priestly leisure class and the philosophers
    - A more 'pure' origin

The Era and the Sources

- Most sources are written in hieroglyphics. This rendered them unreadable for a long time.
  - Napoleons 1799 expedition to Egypt uncovered the Rosetta stone – allowing us to finally translate hieroglyphics
  - By 1822, the french scholar Jean-François Champollion announced a solid translation.
    - By 1832, he published the beginnings of a dictionary right before he died.

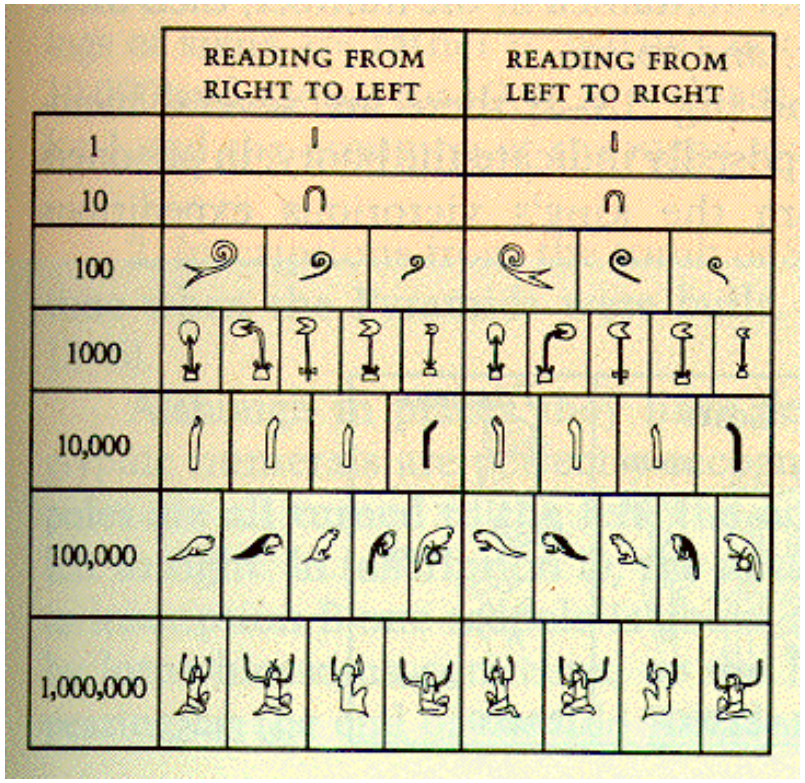
The Era and the Sources

- In 1858, Henry Rhind of Scotland purchased a papryus in Luxor – the Ahmes Paprys
  - Made in 1650 BCE, copied material from the Middle Kingdom (2000 BCE - 1800 BCE)
  - Written in hieratic script
  - Approximately 1 ft high and 18ft long
  - Other papryi have been found, but are similar to the Ahmes papryus in substance



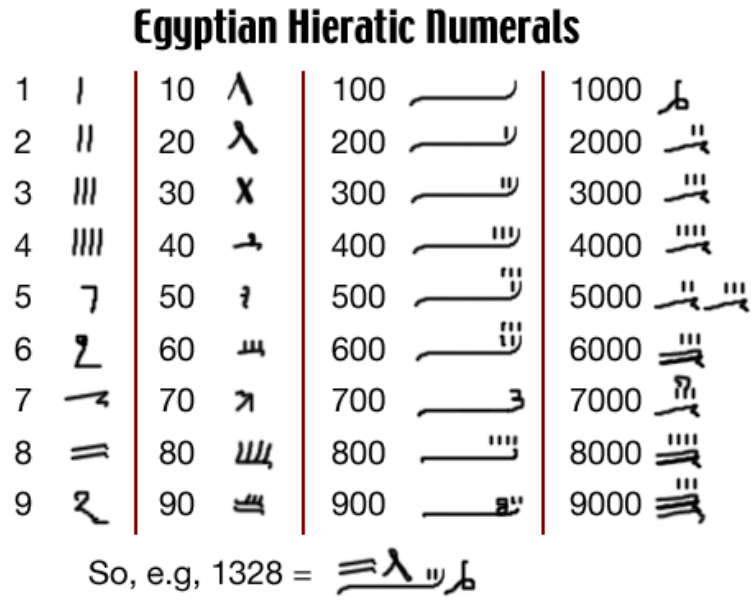
Hieroglyphics

- Hieroglyphic numeration
  - Dates back to 3000 BCE and uses Base 10



Hieratic Script

- Hieratic Script was essentially 'cursive' hieroglyphics
  - Dates back to 2000 BCE and was used by Ahmes
  - Introduced compact 'cipher' notation



Fractions

- In hieroglyphics, unit fractions are written with an oval over the number
- In hieratic script, a dot replaced the oval
- Knowledge of the general fraction doesn't appear in any sources. Mainly unit fractions and their conjugates like $\frac{2}{3}$
  - Working knowledge of identities like

$$\frac{2}{3} \cdot \frac{1}{p} = \frac{1}{2p} + \frac{1}{6p} \qquad \text{and} \qquad 2 \cdot \frac{1}{2p} = \frac{1}{p}.$$

- A fraction was 'reduced' if it was decomposed as a sum of unit fractions and conjugates unit fractions

$$\frac{3}{5} = \frac{1}{3} + \frac{1}{5} + \frac{1}{15}.$$

Ahmes Papyrus (revisited)

- Opened with a table of $\frac{2}{n}$ reductions for odd n from 5 to 101.
  - Examples:

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{15}, \qquad \frac{2}{11} = \frac{1}{6} + \frac{1}{66}, \qquad \frac{2}{101} = \frac{1}{101} + \frac{1}{202} + \frac{1}{303} + \frac{1}{606}$$

- For unknown reasons, used the identity $\frac{2}{n} = \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}$ and not $\frac{2}{n} = \frac{1}{n} + \frac{1}{n}$.
- The papyrus had a similar table for $\frac{n}{10}$

Arithmetic Operations

- Ahmes Papyrus contained 84 problems and solutions
- Multiplication was conducted via successive doubling, or 'duplation'.
  - Example: $69 \cdot 11 = ?$

$$69 + 69 = 138$$

$$(69 + 69) + (69 + 69) = 276$$

$$((69 + 69) + (69 + 69)) + ((69 + 69) + (69 + 69)) = 552$$

Since $11 = 8 + 2 + 1$,

$$69 \cdot 11 = 8 \cdot 69 + 2 \cdot 69 + 69 = 552 + 138 + 69 = 759.$$

- Division was done similarly, but they doubled the divisor instead

Bread and Beer

- Ahmes Papyrus had many 'bread and beer' problems
- **Problem 72:** How many loaves of bread of 'strength' 45 are equivalent to 100 loaves of 'strength' 10 bread?
  - 'strength' or *pesu* is the quotient of the number of loaves by the amount of grain used
- The solution given was simply: $\frac{100}{10} \times 45$.
  - There was no description of how this answer was obtained
- Shows some working knowledge of proportions and ratios

"Heap" Problems

- Some problems in the Ahmes papyrus were 'algebra' in nature, involving abstract, unknown quantities called *heap*
  - For example, some problems called for solving linear equations of the form

$$x + ax = b \qquad \text{and} \qquad x + ax + bx = c.$$

  - The solutions given were different from how we'd solve them today. They used proportions and the "method of false position"

Method of False Position

Problem 24: What is heap if heap and $\frac{1}{7}$ of heap is 19?
Solution: Equivalently, we are solving the equation $x + \frac{1}{7}x = 19$ for x. The "method of false position" is employed by *supposing* heap is 7. In this manner,

$$x + \frac{1}{7}x = 7 + \frac{1}{7} \cdot 7 = 8.$$

Clearly not 19. But, Ahmes writes, since

$$8\left(2 + \frac{1}{4} + \frac{1}{8}\right) = 19,$$

the answer must be

$$\text{heap} = 7\left(2 + \frac{1}{4} + \frac{1}{8}\right) = 16 + \frac{1}{2} + \frac{1}{8}.$$

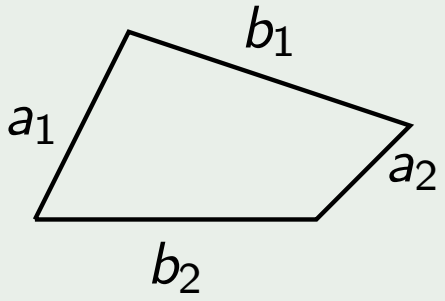
Geometric Problems

- Recall that both Herodotus and Aristotle claimed that geometry came to Greece through Egypt.
- The Ahmes Papyrus did have some geometry in it, but not a lot
  - Other artifacts from Egypt showcase some
- A deed from Edfu dated circa 50 BCE has what we would call *theorems* concerning geometric objects

Geometric Problems

Theorem

Given a quadrilateral of the form



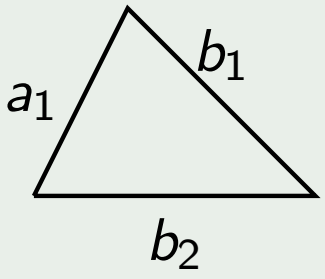
Its area is given by $\frac{a_1+a_2}{2} \cdot \frac{b_1+b_2}{2}$.

- This is true for squares and rectangles, but fails for more exotic quadrilaterals like trapezoids

Geometric Problems

Corollary

Given a triangle of the form

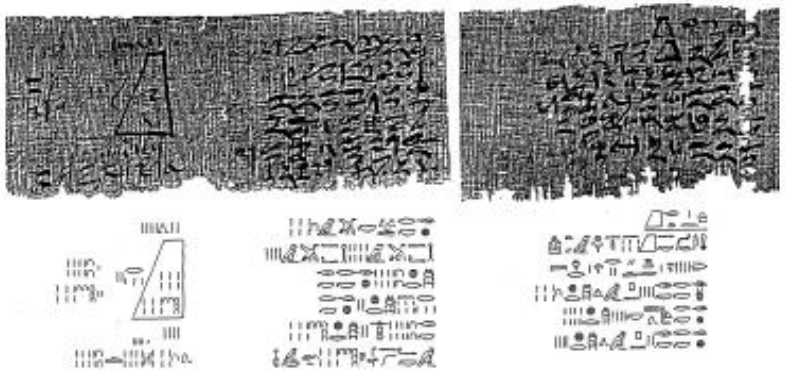


Its area is given by $\frac{a_1}{2} \cdot \frac{b_1+b_2}{2}$.

- This corollary was deduced from the previous theorem by letting $a_2 = 0$. An incredible instance in history where a culture used zero as a concept in magnitude!

Frustrum of Pyramids

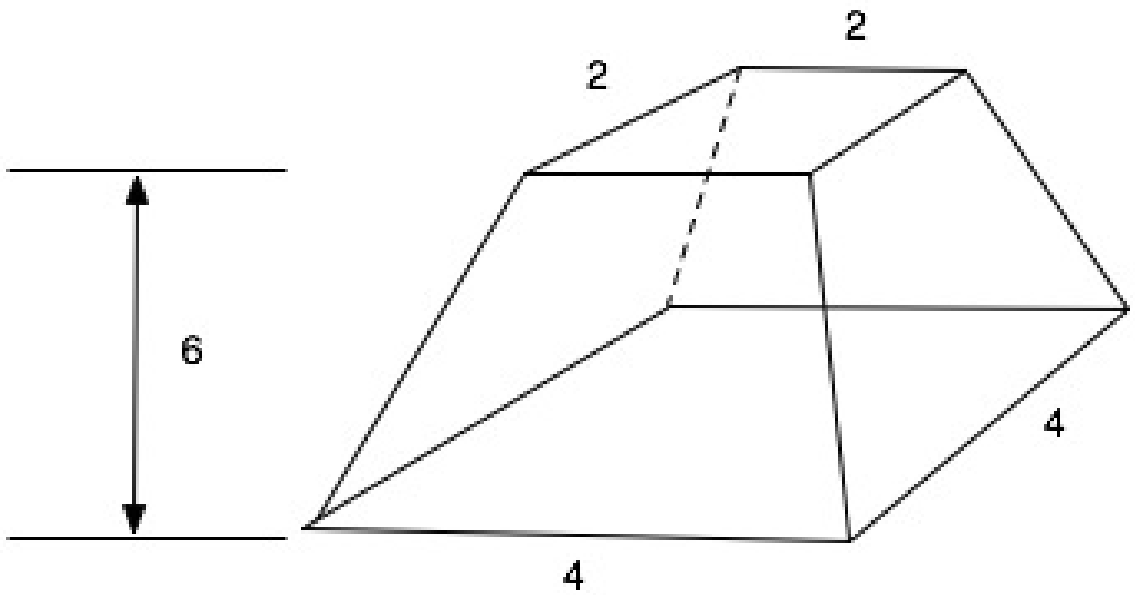
- In true Egyptian fashion, Egyptian geometry concerned itself with square pyramids
- In a papyrus helds in Moscow, there are problems involving the volume of a frustrum of a square pyramid.



A copy of Problem 14.

Problem 14

Problem 14 in the Moscow papyrus asks for the volume of the following object:



Problem 14

- The scribe provides the following solution:
  - The papyrus then solves it: ""If you are told: a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top: You are to square the 4; result 16. You are to double 4; result 8. You are to square this 2; result 4. You are to add the 16 and the 8 and the 4; result 28. You are to take $\frac{1}{3}$ of 6; result 2. You are to take 28 twice; result 56. See, it is of 56. You will find [it] right"
- Amazingly, 56 is the correct answer. The modern formula for the volume of a frustrum of a square pradmid with height h and base lengths a and b is given by

$$V = \frac{h}{3}(a^2 + ab + b^2).$$

- In this case, with $h = 6$, $a = 2$ and $b = 4$, one yields $V = 56$!

Slope Problems

- The construction of a pyramid necessitates a uniform slope
  - Egyptians used the word 'seqt' and instead computed it via $\frac{\text{run}}{\text{rise}}$
  - Vertical units of length were *cubits* or *ells*
  - Horizontal units of length were *hands* where cubit is seven hands
  - Therefore seqt was measured in hands per cubit or hands per ell

Slope Problems

Problem 56 Find the seqt of a pyramid 250 ells high with a square base 360 ells on a side

- The scribe computed

$$\frac{360}{2} \cdot \frac{1}{250} = \frac{1}{2} + \frac{1}{5} + \frac{1}{50} \text{ ells.}$$

- Since seqt is given as hands per ells, he then multiplied the above by 7 and found that the seqt was $5 + \frac{1}{25}$ hands per ell.

Conclusion

- Egyptian knowledge of mathematics seems to be of a practical nature
  - The papyri we have as sources primarily consider concrete questions and are mainly calculations
- Their arithmetic left lots to be desired and their algebra indicates only a start in the direction of the abstract
- The geometry results vary heavily but are clearly influenced by practical needs – slopes and pyramids
  - Sometimes their results are impressive, other times they are just plain incorrect
  - No development of formalization, proofs, or rigor – only computation and trial and error

Conclusion

- In the 1997 book *Numerology or, what Pythagoras wrought*, mathematician Dudley Underwood notes that Egyptian mathematics was "... clever but very, very primitive. With no algebra, no trigonometry, hardly any geometry, and laborious arithmetic, the Egyptians were not going to make any deep mathematical discoveries and in fact they did not."
  - With so little surviving sources, such a statement becomes difficult to believe
  - Even *with* the data we do have, it is hard to say that Egyptian mathematics had "no algebra" and "hardly any geometry"
  - While their surviving discoveries aren't deep compared to the cultures that came after them, it is unfair to not appreciate the care and precision with which they tackled the practical arithmetic and geometric problems of their time