Early Greek Mathematics: Thales and Pythagoras

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Outline

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The Era and the Sources

- Although the Seleucid period is Mesopotamia occurred around 300 BCE, a shift in the intellectual leadership of the world occurred around 800 BCE from Mesopotamia and Egypt to the Mediterranean Sea
  - Sometimes this shift is referred to as the start of the Thalassic period (sea period)
  - First Olympic games held in 776 BCE
  - Unfortunately, there are almost no remaining primary sources from Greece between 800 and 500 BCE
    - It is thought this is due to the prevalence of oral tradition.

Mathematics

- By the sixth century BCE (500s BCE), two men seem to have appeared on the mathematical scene: Thales and Pythagoras
  - Unfortunately, there does not exist any primary source data for their achievements
  - The earliest reliable sources on Greek mathematics come in the form of persistent tradition and third-hand sources
    - For example, in the fourth century BCE, Plato and his students began recording the mathematical accomplishments of Thales and Pythagoras
    - Similarly, Eudemus of Rhodes (a student of Aristotle) was a historian of science c. 350 BCE. He documented much of mathematics history up until then
  - Unfortunately, even these recordings have been lost and only summaries of these recordings exist in sources.
    - As a result, we have to rely on existing sources like writings due to Proclus Lycaeus in 400 CE when he summarized the writings of Eudemus

Mathematics

- It would take until Plato and Aristotle in the fourth century BCE until we have access to good, secondary accounts of Greek mathematics
- In particular, after Thales and Pythagoes, the big names of the Medetaranean were:
  - Archytas of Tarentum (428 BCE)
  - Hippasus of Metapontum (400 BCE)
  - Democritus of Thrace (460 BCE)
  - Hippias of Elis (460 BCE)
  - Hippocrates of Chios (430 BCE)
  - Anaxagoras of Clazomenae (428 BCE)
  - Zeno of Elea (450 BCE)

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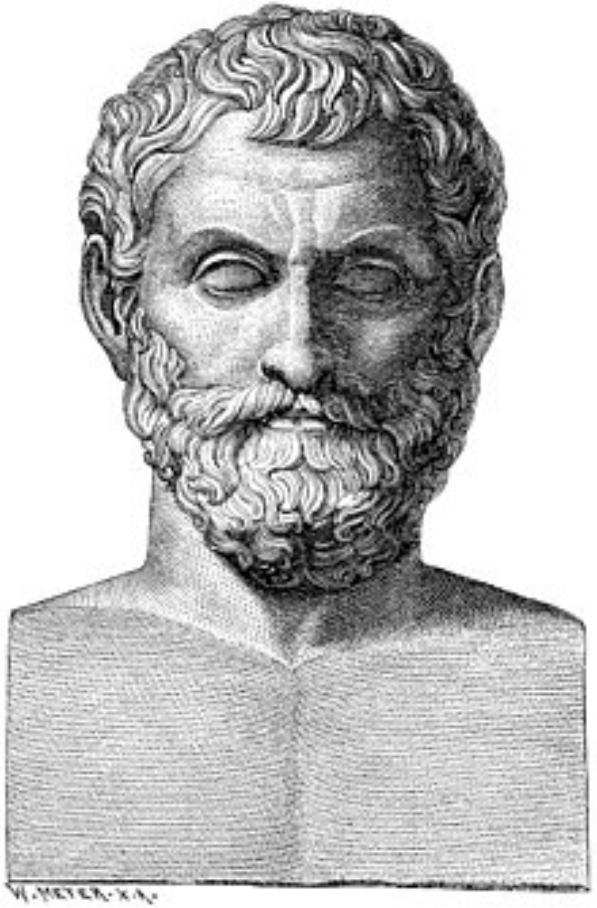
Overview

- Origins of Greek mathematics are usually attributed to the Ionian and Pythagorean schools
  - Ionian school led by Thales of Miletus (c. 624-548 BCE)
  - Pythagorean school led by Pythagoras of Samos (c. 580-500 BCE)
- Both individuals and their schools were located on the Mediterranean sea
  - Ample opportunity to travel to existing centers of learning
    - Reportedly, both traveled to Egypt and Babylon where they picked up geometry, astronomy, and other mathematical discoveries.

Thales



Thales of Miletus (c. 624 - 548 BCE)

Thales

- Existing sources indicate he was exceptionally clever and regard him as the first philosopher
- Given his clear travels to Babylon, the 'Theorem of Thales' may well have been passed to him by the Babylonians
  - Tradition purports that he was originator of the deductive organization of geometry
    - Some refer to him as the first 'true' mathematician
  - Supposedly he provided a 'demonstration' of the theorem and thus earned the name of the theorem
- In addition to this theorem, tradition has attributed four other theorems to Thales

Thales' Theorems

(1) A circle is bisected by a diameter.

(2) The base angles of an isosceles triangle are equal.

(3) The pairs of vertical angles formed by two intersecting lines are equal.

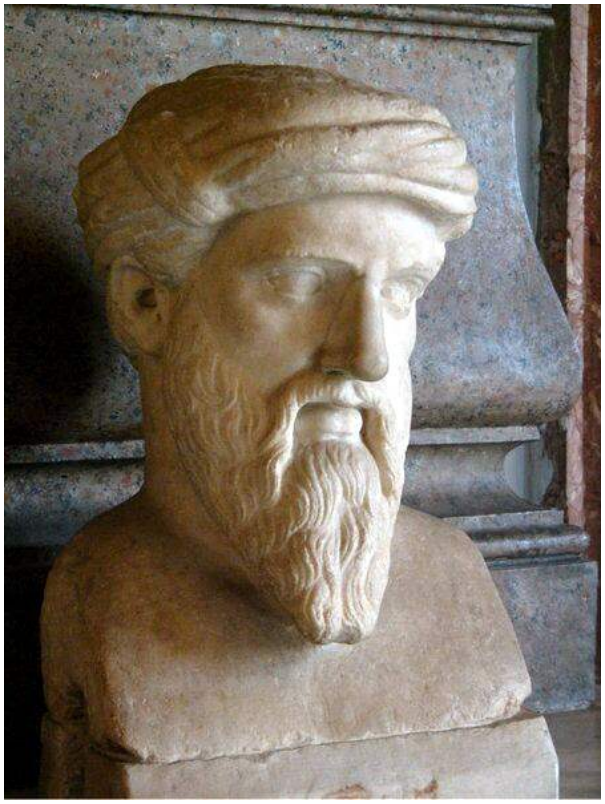
(4) If two triangles are such that two angles and a side of one are equal, respectively, to two angles and a side of the other, then the triangles are congruent.

- Unfortunately, there are no documents that support the claim that Thales established these theorems.

Thales' Theorems

- The closest we have in terms of establishing Thales' ownership is the following:
  - A student of Aristotle, Eudemus of Rhodes (c. 320 BCE) wrote a history of mathematics, which was lost, but summarized in part by someone later.
  - In 500 CE, this partial summary was incorporated into the philosopher Proclus' *Commentary on the First Book of Euclids Elements*.
- Despite this weak sourcing, modern historians agree that Thales was the first many in history to whom specific mathematical observations are attributed
  - That the Greeks were the first to integrate logical structure into geometry is not a contested fact. That Thales alone took this step is another story.

Pythagoras



Pythagoras of Samos (c. 580-500 BCE)

Pythagoras

- Born in Samos – not far from Miletus (birthplace of Thales)
- While some historians have theorized that Pythagoras was a student of Thales, this seems unlikely due to the 50 year age gap
- Traveled often in his youth Through his travels, not only did he encounter mathematics but he also embraced the religions of the world
  - Returned to Greece and settled in Croton (in modern day Italy)
  - Established a secret society founded on mathematics and philosophy
  - Embraced the religions of the world and remained a devout mystic

Pythagoras

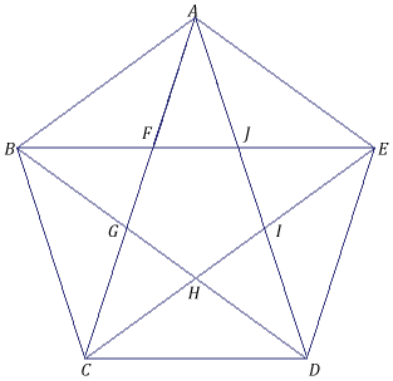
- Unfortunately, sources for his mathematics are just as scarce as Thales, but made worse by the secret nature of his society
  - It is agreed that most of 'his' discoveries were done by the society, since it was customary to attribute discoveries to the leader
- The Pythagorean Order maintained that philosophical and mathematical studies were the moral basis for life
- While Thales may have started the discussion, the Pythagoreans are attributed the first real treatment of the philosophy of the principles of mathematics
  - Unlike Egyptians and Babylonians, the Pythagoreans were less concerned with the practical needs of mathematics
  - Their influence was far an deep within the Greek world

Pythagorean School

- The motto of the Pythagorean school is said to have been "All in number."
- It is believed that the school was heavily influenced by Babylonian mathematics
  - Even the Pythagorean theorem is probably Babylonian
  - Some claim that the Pythagoreans deserve the name of the theorem since they were the first to 'demonstrate' it
    - There is no evidence to support this.
- A special symbol for the school was the 5-pointed star: 
  - Of interest is that this symbol also existed in Babylonian art

Pythagorean School

- An interesting finding of their is the following: Inscribe the 5-pointed star in a regular pentagon (a so-called pentagram):



- There is a large amount of symmetry in his figure. In particular, the Pythagoreans deduced that

$$\frac{BD}{BH} = \frac{BH}{HD}.$$

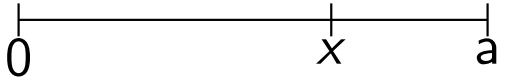
- This ratio would later be called the *golden ratio* ϕ

Golden Ratio

- Greeks were quite familiar with this ratio. Eventually, instead of writing "the division of a segment in mean and extreme ratio", they just referred to it as "the section"
- Several thousand years later, Johannes Kepler would write:
  - "Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."
- While it seems clear that the Greeks were aware of how to divide a line along the golden ratio, were they aware of the value?

Golden Ratio

- One construction of the golden ratio is as follows:
  - Consider



  - This is constructed specifically so that

$$\frac{a}{x} = \frac{x}{a-x} \iff a^2 - ax = x^2 \iff x^2 + ax = a^2$$

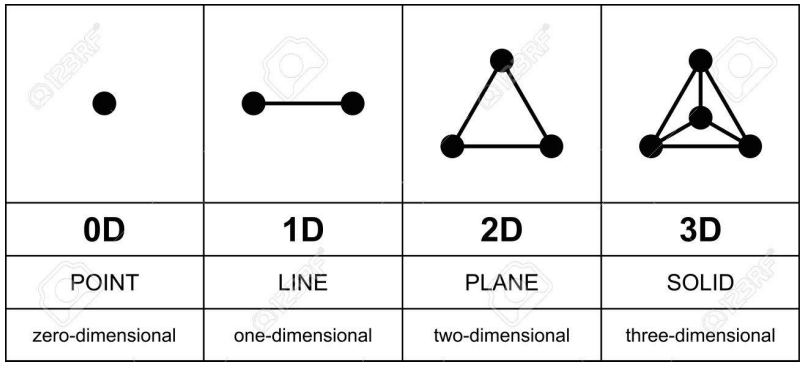
  - The Babylonians approximated these types of quadratics easily, so it is likely the Pythagoreans could have as well
- It is highly likely that, at the very least, the Pythagoreans were aware of the geometric construction of such a point
  - The known construction is nowadays referenced from Euclids elements

Number Mysticism

- Pythagorean Order were obsessed with numbers and attached additional meaning to each one
  - One could go as far as saying they worshiped numbers
- 7 was special due to the seven wandering stars/planets of the time
- Odd numbers were considered 'male' and even 'female'
- For integers:
  - 1 is the generators of all numbers and is the number of reason
  - 2 is the first female number and the number of opinion
  - 3 is the first male number and the number of harmony
  - 4 is the number of justice and retribution
  - 5 is the number of marriage as the union of the first male and female numbers
  - 6 is the number of creation

The Number 10

- The number 10 was considered the number of the universe.
- In particular, the Pythagoreans worshiped the number 10 since it was the sum of all possible geometric dimensions



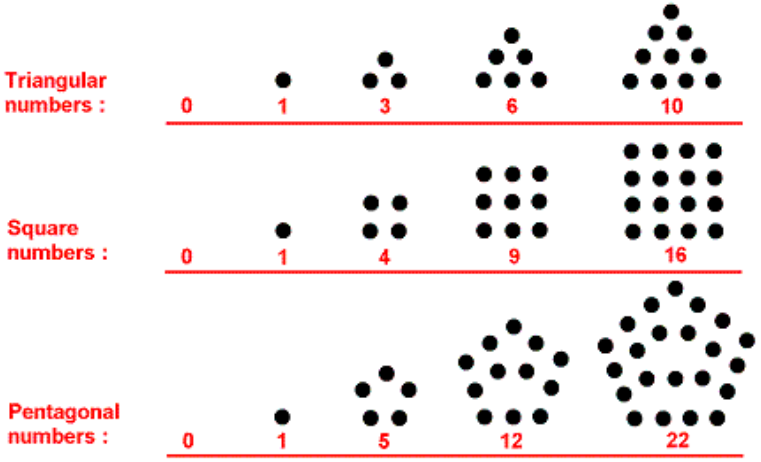
0D	1D	2D	3D
POINT	LINE	PLANE	SOLID
zero-dimensional	one-dimensional	two-dimensional	three-dimensional

- $1 + 2 + 3 + 4 = 10$
  - Of interest is that their love of 10 was not due to human anatomy

Arithmetic

- Recall that in Egypt, there were natural numbers and then unit fractions
- In Babylon, the considered just rational fractions of natural numbers
- To the Pythagoreans, "number" implied integers and they viewed fractions as a ratio or relationship on two natural numbers
  - They emphasized the theoretical number concept and deemphasized numbers as merely a computational tool
  - Arithmetic for the Pythagoreans became an intellectual tool as well as a problem solving technique
- Pythagoreans did a brilliant job of linking arithmetic to geometry and deduced many general formulas for 'triangle numbers', 'square numbers', 'pentagonal numbers', etc.

Polygonal Numbers



- Triangle Numbers: $N = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
- Square Numbers: $N = 1 + 3 + 5 + \ldots + (2n - 1) = n^2$
- Pentagonal Numbers:
  $N = 1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n-1)}{2}$

Polygonal Numbers

- Reportedly, these type of observations are what led Philolaus (a later Pythagorean c. 390 BCE) to claim that
  - "All things which can be known have number; for it is not possible that without number anything can be either conceived or known."
- This bold statement embodies the Pythagorean ideology.
- Other discoveries of Pythagoras along these lines was the mathematics behind music harmony and rational notes
  - Of interest was their attempt to extend the harmony of musical notes to celestial bodies, claiming that they emitted harmonious tones dubbed the "harmony of the spheres"
  - These mystical extrapolations of their mathematical discoveries were common for the overly zealous Pythagoreans

Proportions

- The philosopher Proclus (c. 400 BCE), likely quoting Eudemus, attributed two mathematical discoveries to Pythagoras:
  (1) Construction of the regular solids
  (2) Theory of proportionals
- Reportedly, when Pythagoras traveled to Mesopotamia, he learned of the three means of the world: Given two values a and b,
  - Arithmetic mean: $\frac{a+b}{2}$
  - Geometric mean: $\sqrt{ab}$
  - Subcontrary or harmonic mean: $\frac{1}{\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right)}$
    - Simply the reciprocal of the average of the reciprocals
    - Upon simplification, equal to $\frac{2ab}{a+b}$

Proportions

- At some point, the Pythagoreans generalized this theory greatly. If b is the 'mean' of a and c where $a < c$, then a, b, and c are related in one of the following ways:

1. $\frac{b-a}{c-b} = \frac{a}{a}$
2. $\frac{b-a}{c-b} = \frac{a}{b}$
3. $\frac{b-a}{c-b} = \frac{a}{c}$
4. $\frac{b-a}{c-b} = \frac{c}{a}$
5. $\frac{b-a}{c-b} = \frac{b}{a}$
6. $\frac{b-a}{c-b} = \frac{c}{b}$
7. $\frac{c-a}{b-a} = \frac{c}{a}$
8. $\frac{c-a}{c-b} = \frac{c}{a}$
9. $\frac{c-a}{b-a} = \frac{b}{a}$
10. $\frac{c-a}{c-b} = \frac{b}{a}$

Proportions

- Observe that
  - 1.) $\frac{b-a}{c-b} = \frac{a}{a} \implies b = \frac{a+c}{2} =$ arithmetic mean
  - 2.) $\frac{b-a}{c-b} = \frac{a}{b} \implies b = \sqrt{ab} =$ geometric mean
  - 3.) $\frac{b-a}{c-b} = \frac{a}{c} \implies b = \frac{2ac}{a+c} =$ harmonic mean
- So this list of 10 relations on means was a true generalization of the existing means of the time

Other Pythagorean Contributions

- At some point the Pythagoreans began investigating 'odd-odd' and 'even-odd' numbers
  - Based on whether the number was the product of two odd numbers or an even and an odd
  - Because of this, sometimes 'even number' was reserved for powers of 2
- By the time of Philolaus, the notions of primes and coprimes were known and embraced by the Pythagoreans.
  - Speusippus, a nephew of Plato, asserted that 10 was 'perfect' for the Pythagoreans because it is the smallest integer n for which there are just as many primes and nonprimes between 1 and n

Other Pythagorean Contributions

- Additionally, the Pythagoreans developed notions of 'perfect', 'abundant', and 'deficient' numbers
  - Dependent on whether the sum of the proper divisors is equal to, greater than, or less than the number itself.
- Furthering the categorization of numbers based on their divisors, Pythagoreans later developed the notion of 'amicable' numbers
  - Two integers a and b are said to be 'amicable' if a is the sum of the proper divisors of b and vice versa
  - The smallest such pair are 220 and 284

Pythagorean Triples

- Their general formula for Pythagorean Triples was given by

$$\frac{m^2-1}{2}, \qquad m, \qquad \text{and} \qquad \frac{m^2+1}{2}$$

where m is an odd integer.

  - Recall that it is theorized the Babylonians used $p^2 - q^2$, $2pq$, and $p^2 + q^2$ for $p > q$
  - Letting $p = m$ and $q = 1$, we yield the Pythagorean formulae
  - Of interest is that the Pythagorean formula did not catch all Pythagorean triples, only the ones with m odd
    - For example, $(8, 15, 17)$ is a triple unobtainable by the Pythagorean formula, but achieved by letting $p = 4$ and $q = 1$

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General Greek Numeration

- If the great worked due to Thales and Pythagoras couldn't stand the test of time, it is impossible for us to understand the mathematics used by everyday laymen and traders
  - What we can try to understand, however, is the numeration they used to write their numbers
  - Early Greek numeration fell into two types:
    - Earlier Attic notation (sometimes called Herodianic)
    - Later Ionian system (sometimes called alphabetic)
  - Both systems were base 10

Attic Numeration

- A simple iterative scheme like that of Egyption hieroglyphics
- Survived in ancient Greece until the earlier Alexandrian Age (c. 300 BCE)
- The system went as follows:
  - Numbers 1-4 were given by vertical strokes
  - 5 was given by Π – the first letter of 'pente'
    - Sometimes written Γ
    - Note: At this time, lowercase letters did not exist.
  - Numbers 6-9 are given by juxtaposition of Γ and vertical strokes
    - $\Gamma||| = 8$

Attic Numeration

- For numbers greater than or equal to 10, symbols were reserved for multiples of 10
  - Δ = 'deka' = 10
  - H = 'hekaton' = 100
  - X = 'khilioi = 1000
  - M = 'myrioi = 10000
- Some combinations were in use: for multiples of 5, they'd use Γ and hide a smaller symbol inside the hook to denote the multiple
  - Example:

$$= 10{,}000 + 5000 + 1000 + 50 + 20 + 3$$
$$= 16{,}073.$$

Ionian System

- By about 300 BCE, that Attic system gave way to the Ionian or alphabetic system
- This system used the letters of the Greek alphabet and associated to each letter one of the numbers from $1, 2, \ldots, 9, 10, 20, \ldots, 90, 100, 200, \ldots, 900$
  - This system required 27 unique symbols
  - Problem: The Greek alphabet at the time only had 24. Thus, three letters were added: digamma, koppa, and sampi

α	β	γ	δ	ε	ϝ	ζ	η	θ
1	2	3	4	5	6	7	8	9
ι	κ	λ	μ	ν	ξ	ο	π	ϙ
10	20	30	40	50	60	70	80	90
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
100	200	300	400	500	600	700	800	900

Greek Alphabetic Numerals

Ionian System

- For numbers greater than 1000, a lower stroke was added beforehand
  - Example: $,\theta = 9000$
- While the Greeks were aware of the positional notation of Babylon, they seemingly chose not to use it
- For fractions, they seemed to prefer the unit fractions of Egypt
  - They used an accent after the number to denote the reciprocal
    - Example: $\lambda\delta' = \frac{1}{34}$
  - Unfortunately, without a full positional system, this could be confused with $30\frac{1}{4}$
    - It seems the Greeks relied on context to resolve any ambiguity

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Logistic and Conclusions

- Unfortunately, many of the sources we have from Greece between 600 and 450 BCE are lost.
  - What sources we do have rely heavily on tradition and third-hand summarizations
  - As a result, it is impossible to get a true picture of how Greek layman and mathematicians truly conducted themselves – we can only do our best to paint a picture

Logistic and Conclusions

- It is up to us how much we believe they created themselves, how much was borrowed from Egypt and Babylon, and how much has been embellished
- Most historians, at the very least, agree that Thales and the Pythagoreans were a cut above the Egyptians and Babylonians and both made great advances in the abstraction of mathematics
- In particular, they both helped formulate mathematics into an intellectual discipline as well as a practical one
  - Interestingly, the Pythagorean Order did consider the practical side of mathematics, but relegated it to a sub-discipline called 'logistic'

Conclusions

- For the most part, Thales and Pythagoras set the scene for future Greek mathematicians
  - By $450BCE$, many Greek mathematicians would crop up and begin further solidifying the notions of deductive reasoning, geometry, and the infinite
  - Fortunately, due to the works of Plato and Aristotle in the fourth century, the mathematics done by these men are far better documented than those of Thales and Pythagoras