Research Statement

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Overview

My research interests are interdisciplinary: mathematics and history. Historically, I am interested in the various effects that war has on the mathematical curriculum. Specifically, I am interested in World War II and how it changed the American undergraduate mathematics curriculum during the Cold War era. Mathematically, I am interested in functional analysis and extending classical function theoretic results from the disk algebra to uniform algebras. Additionally, both research interests have perfectly viable paths for undergraduate investigations as well..

Mathematical Research

Overview of Research Interests

Within functional analysis, oftentimes natural questions are posed about a particular algebra of functions such as the algebra of functions bounded and holomorphic on a planar domain or bordered Riemann surface. A modern approach to such problems involves studying the representations of the algebra as bounded linear operators on a particular Hilbert space. In this manner, one has access to a host of new Hilbert space properties and techniques such as orthogonality, inner products, and reproducing kernels. These objects and their associated operators (such as those due to Toeplitz) are useful in their applications to image processing and data science.

A classical example of this is the disk algebra. Let $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ be the open disk in the complex plane and recall that a holomorphic function is one that is complex differentiable in a neighborhood of every point. The disk algebra, denoted $H^\infty(\mathbb{D})$, is the algebra of functions on $\mathbb{D}$ that are both bounded and holomorphic. This naturally occurring algebra of functions has been the source of many problems within the last century. Now, let $H^2(\mathbb{D})$ denote those holomorphic functions defined on $\mathbb{D}$ with square-summable power series. That is, if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ for $z \in \mathbb{D}$, then $\sum_{n=1}^{\infty} |a_n| < \infty$. This space turns out to be Hilbert space whose inner product matches that coming from the space L^2. Moreover, it turns out that the algebra $H^\infty(\mathbb{D})$ is isometrically isomorphic to the space of bounded linear operators on $H^2(\mathbb{D})$. In this manner, recent developments of problems dealing with H^∞ have been solved by first porting the question over to H^2 and then using Hilbert space techniques to solve the problem. One example is the classical Nevanlinna-Pick interpolation problem, which was solved in the early 20th century by purely function theoretic methods, but has taken on a new life via the injection of Hilbert space methods, which has allowed for numerous extensions and generalizations. [2]

Of particular interest to my research is taking classical settings like the disk algebra $H^\infty(\mathbb{D})$ and imposing some additional constraints. As the simplest example, consider functions in H^∞ that are forced to agree at two prescribed points a and b in $\mathbb{D}$. This constrained subalgebra, denoted $\mathscr{A}_{a,b}$, is of particular interest in its relation to construction of general finite-codimensional subalgebras of uniform algebras. It is natural to ask the same classical questions that exist for $H^\infty(\mathbb{D})$, but about $\mathscr{A}_{a,b}$ instead. While most theorems do carry over, their proofs require passing to more than just a single representation-carrying Hilbert space. In fact, one has to consider an infinite family of representations. Wrestling with this situation serves as a launching point for many developments in this area of functional analysis and it is here that my research interests lie.

Particular Research Problems and Recent Work

The classical problems I am interested in extending are those due to Widom and Szegő.[1] Additionally I am interested in extending recent work on Nevanlinna-Pick interpolation problems for particular subsets of the complex bidisk called distinguished varieties.[2] The classical versions of these are established for the unit disk $\mathbb{D}$ and the associated algebra $H^\infty(\mathbb{D})$ – bounded, holomorphic functions defined on $\mathbb{D}$. Thus, the classical versions only make use of one reproducing kernel Hilbert space that carries a representation for $H^\infty(\mathbb{D})$, namely the Hardy space H^2. It turns out, however, once one changes the setting away from $\mathbb{D}$, the associated version of the theorem or problem warrants the consideration of not just one representation Hilbert space, but an entire family of them. The exact choice of family depends on the setting, but passing to it allows for classical results to be extended and established for a wide variety of settings. For example, recent work of mine involves establishing a Widom type theorem for finite codimensional subalgebras of various classes of uniform algebras.

To discuss the classical Widom theorem on the unit disk $\mathbb{D}$, one needs to define Toeplitz operators. For a fixed $\phi \in L^\infty(\mathbb{T})$, define the multiplication operator $M_\phi \colon L^2 \to L^2$ by $M_\phi f = \phi f$. Let $V \colon H^2 \to L^2$ be the inclusion map and define the Toeplitz operator with symbol ϕ by $T_\phi = V^* M_\phi V$. The classical Widom theorem says: Given a unimodular $\phi \in L^\infty$, there exists $f \in H^\infty$ such that $\|f - \phi\| < 1$ if and only if T_ϕ is left invertible [4]. The theorem has applications in the theory of stationary Gaussian processes.

Over the years, versions of the Widom theorem have been established for spaces other than the unit disk. In 1974, Abrahamse proved a version for multiply connected domains [1]. More recently, in 2017, Balasubramanian, McCullough, and Wijisooriya established a version for the Neil Algebra [3] . The Neil Algebra is a constrained subalgebra of the $H^\infty(\mathbb{D})$:

$$\mathfrak{A} = \{f \in H^\infty(\mathbb{D}) \ : \ f'(0) = 0\}.$$

These versions of the Widom theorem rely heavily on the use of a family of reproducing kernel Hilbert spaces that each carry a representation of the associated algebra. In the case of the Neil algebra, these spaces are given by the following: For $\alpha = (\alpha_1, \alpha_2)$ in the compact 2-sphere $\mathbb{S}^2$,

$$H^2_\alpha = \{f \in H^2 \ : \ f(0)\alpha_2 = f'(0)\alpha_1\}.$$

Now, for a fixed $\phi \in L^\infty(\mathbb{T})$, we can define a Toeplitz operator on each of these representation spaces. For each $\alpha \in \mathbb{S}^2$, let $V_\alpha \colon H^2_\alpha \to L^2$ be the inclusion map. Let $T^\alpha_\phi \colon H^2_\alpha \to H^2_\alpha$ be the map defined by $T^\alpha_\phi = V^*_\alpha M_\phi V_\alpha$. T^α_ϕ is then the Toeplitz operator with symbol ϕ with respect to α. The Widom theorem for the Neil algebra says: Given a unimodular $\phi \in L^\infty$, there exists $f \in \mathfrak{A}$ such that $\|f - \phi\| < 1$ if and only if T^α_ϕ is left invertible for each $\alpha \in \mathbb{S}^2$.

[1] My work in extending Szegő's theorem stems from work done in [3] where the authors formulate a version for the Neil Algebra. My work endeavors to extend this theorem to a variety of broad classes of uniform algebras.

[2] My work in this avenue involves extending and building upon the work done in [6].

In recent joint work with Michael Jury, we established a version of the Widom thoerem for a different constrained subalgebra of $H^\infty(\mathbb{D})$, namely

$$\mathscr{A}_{a,b} = \{f \in H^\infty(\mathbb{D}) \ : \ f(a) = f(b)\}.$$

In this case, the reproducing kernel Hilbert spaces that carry representations for $\mathscr{A}_{a,b}$ are given by

$$H^2_t = \{f \in H^2 \ : \ f(a) = tf(b)\},$$

where t is taken from the compact Riemann sphere $\mathbb{C} \cup \{\infty\}$. Defining the Toeplitz operators T^t_ϕ for each H^2_t in the natural way, a Widom theorem for $\mathscr{A}_{a,b}$ arises: Given a unimodular $\phi \in L^\infty$, there exists $f \in \mathscr{A}_{a,b}$ such that $\|f - \phi\| < 1$ if and only if T^t_ϕ is left invertible for each $t \in \mathbb{C} \cup \{\infty\}$. Thus, when passing to a constrained subalgebra of $H^\infty(\mathbb{D})$, it becomes necessary to consider an entire family of Toeplitz operators.

Current and Future Work

In current joint work with Michael Jury, we establish (among other things) a similarly flavored Widom theorem for a broad class of finite-codimensional subalgebras of uniform algebras. With some light assumptions imposed on a given uniform algebra A, a theorem due to Gamelin shows that a finite-codimensional subalgebra $\mathscr{A} \subseteq A$ is obtained by iteratively either vanishing a derivative to a finite order at a point, or by asking the functions to agree at two points. Further, Gamelin shows that the algebra $\mathscr{A}$ can be identified with a constrained subalgebra of the algebra of bounded functions on a finite, connected Riemann surface. In this setup, $\mathscr{A}$ is then seen as a hypo-Dirichlet algebra and thus enjoys a rich structure [5].

As with the algebras $\mathfrak{A}$ and $\mathscr{A}_{a,b}$, we find that there is again a family of reproducing kernel Hilbert spaces, denoted $H^2_{\alpha,Q}$, such that each one carries a representation for $\mathscr{A}$. (Here $(\alpha, Q) \in \Sigma \times \mathscr{P}$, where Σ and $\mathscr{P}$ are specific compact subsets of $\mathbb{R}^\sigma$ and $M_{\gamma\times\gamma}$ respectively for some fixed, finite σ and γ.) In this setting, one may again show that $\|f - \phi\| < 1$ if and only if each of a suitable family of Toeplitz-like operators, paramaterized by the space $\Sigma \times \mathscr{P}$, is left invertible.

Future work in this direction would see the relaxing of the light assumptions placed on the given uniform algebra in order to yield a stronger, more general Widom type theorem for uniform algebras. Further, a deeper investigation into the intimate connection between our methods and the general theory of hypo-Dirichlet spaces is in order. Aside from the Widom theorem, there are still many viable research problems coming from the consideration of an infinite family of representation-carrying Hilbert spaces. Of interest is establishing Szegő type theorems for different classes of uniform algebras and also refining and improving on current results for Nevanlinna-Pick interpolation on distinguished varieties.

Historical Research

Overview

My specific historical interests are on the effects that war has on mathematical curriculum. In particular, recent interests have had me investigating the effects that World War II had on the American undergraduate mathematics curriculum during the Cold War era.

Remarkably, this pocket of history has received very little academic attention. Historians of American higher education have investigated the influence the war had on neighboring disciplinary curricula like engineering.[3] More broadly, historians have examined the massive impact World War II had on higher education during the early 1940s and into the postwar era[4], and the responses that both higher educational institutions and society had as a result.[5] Historical investigations on the undergraduate mathematical curriculum tends to be broad, sweeping looks at its overall history with no keen eye on the immediate post-war era.

Current Work

In recent work, I focus on a particular facet of World War IIs impact on higher education, namely, the propagation of applied coursework in the mathematics curriculum.

My investigation makes use of various mathematical journals published by the Mathematical Association of America (MAA). In particular, I used the primary journal of the MAA, The American Mathematical Monthly. This journal served as the primary location of discourse amongst mathematics educators in higher education in the early half of the 20th century. Poring through the American Mathematical Monthly, I aimed to gain an understanding of the conversation generated about the undergraduate mathematics curriculum by higher institutional mathematics educators. To assess the extent to which the undergraduate mathematics curriculum changed in response to both war pressures as well as the mathematics educators recommendations, I made use of available course offerings at an array of higher educational institutions during the years 1935 and 1955. [6]

I argue that, among other things, World War II induced a short-term shift in the actual undergraduate mathematics curricula toward more applied coursework. The war caused a fierce, immediate recognition that applied mathematics was imperative to the war effort and the general undergraduate mathematics curriculum underwent radical changes to accommodate the war pressures. However, the permanence of these changes was limited. After the war, more traditional, theoretical mathematics educators in higher education harkened for a return to the prewar,

[3]For studies concerning the curriculum changes for undergraduate engineering and general undergraduate sciences, see Robert E. Horton, Mathematics in the Engineering Curriculum in *Mathematics Magazine*, Vol. 32, No. 3 (Jan.-Feb., 1959) pp. 137-149; Bruce Seely, Research, Engineering, and Science in American Engineering Colleges: 1900-1960 in *Technology and Culture*, Vol. 34, No. 2 (Apr. 1993) pp 344-386

[4]For an overview, see Roger L. Geiger, *The History of American Higher Education: Learning and Culture from the Founding to World War II* (Princeton University Press, Reprint edition, 2016); V.R. Cardozier, *Colleges and Universities in World War II* (Praeger, First Edition edition, 1993); Charles Dorn, Promoting the Public Welfare in Wartime: Stanford University during World War II, in *American Journal of Education*, Vol. 112, No. 1 (Nov. 2005), pp. 103-128; Charles Dorn, A Womans World: The University of California, Berkeley, during the Second World War, in *History of Education Quarterly*, Vol. 48, No. 4 (Nov. 2008), pp. 534-564.

[5]For general studies of higher education in post-World War II America, see, Marvin Lazerson, The Disappointments of Success: Higher Education after World War II, in *The Annals of the American Academy of Political and Social Science*, Vol. 559, The Changing Educational Quality of the Workforce (Sep. 1998) pp. 64-76; Clark Kerr, Higher Education: Paradise Lost?, in *Higher Education*, Vol. 7, No. 3 (Aug, 1978), pp. 261-278; Martin Trow, The Public and Private Lives of Higher Education, in *Daedalus*, Vol. 104, No. 1, American Higher Education: Toward and Uncertain Future, Volume II (Winter, 1975), pp. 113-127.

[6]Obtaining course registers is greatly aided by Walter Meyer et al., Cajori Two: Survey of Undergraduate Mathematics Courses in the 20th Century. Adelphi University. This study gathered the mathematics departmental course registers from twenty various institutions of higher education in America for the years 1905, 1915, . . ., 1995, 2005 and deduced trends about the higher institutional mathematics course offerings across America in the 20th century. In particular, their data consisted of scanned copies of these course registers supplied by archivists at the various institutions. These registers are made freely available at http://matcmp.ncc.edu/taormij/cajori_two/catalog_scans.php and gives me access to an array of mathematics course offerings for the years 1935 and 1955. The twenty available institutions range every geographic region of the United States, cover both public and private institutions as well as consists of colleges and universities. These course offerings allow me to assess the nations higher institutional undergraduate mathematics curriculum before, during and after World War II.

unapplied mathematics curriculum, while the applied mathematics educators in higher education, bolstered by their success in the war, advocated for a more permanent spot in the curriculum. Thus began a fiery debate amongst mathematics educators in higher education about just how applied the mathematics curriculum should be. Both groups believed that mathematics was useful and should be required in a liberal education, but they disagreed on what it meant for mathematics to be useful. Traditional, unapplied mathematics educators argued that mathematics was useful in the creation of a logical, rational citizen and hence led to a society be dominated by deductive reasoning and independent thought. Applied mathematics educators contended that mathematics was useful only when it created citizens who used mathematics to advance of other, practical pursuits and sciences whose physical applications would lead to a scientifically superior society. During the Great Depression, mathematics educators view of mathematical utility aligned with the applied viewpoint that kept the mathematical discipline practical and desirable in a decade where impractical pursuits were perilous. Whereas after the war, as the Cold War drew upon the nation and the demand for rational, scientifically aware citizens grew, a few mathematics educators came to make curricular suggestions that reconciled both the applied and unapplied points of view. Essentially, World War II gave credence to the applied mathematics educators notion of mathematical utility and began a debate amongst all mathematics educators on the larger purpose of mathematics in a liberal education which eventually, in response to Cold War pressures, saw a few educators settle on both an desiring both an applied and pure curricular component.

Future Work

Further investigations in this direction will see a broadening of the source representation to include a better understanding of historically black colleges and womens colleges. These influential institutions simply don't have the correct amount of representation in the American Mathematical Monthly publications of the time. Additionally, a deeper understanding of the nature of 'applied' versus 'unapplied' in the education rhetoric is in need in order to build a better picture of how the educators in higher education viewed their own subject. All in all, these are much needed historical investigations and constitute only a few of the possible research avenues one might take.

References

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[3] S. Balasubramanian, S. McCullough, and U. Wijesooriya. Szego and Widom Theorems for the Neil Algebra. *ArXiv e-prints*, June 2017.

[4] Ronald G. Douglas. *Banach algebra techniques in operator theory*, volume 179 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[5] T. W. Gamelin. Embedding Riemann surfaces in maximal ideal spaces. *J. Functional Analysis*, 2:123–146, 1968.

[6] Michael T. Jury, Greg Knese, and Scott McCullough. Nevanlinna-Pick interpolation on distinguished varieties in the bidisk. *J. Funct. Anal.*, 262(9):3812–3838, 2012.