SOLUTIONS

1. Is the space $\mathbb{R}_\ell$
   (a) Baire?
   (b) Contractible?
   SOLUTION:
   (a) Yes. We show that for every sequence $U_i$ of dense open sets in $\mathbb{R}_\ell$ the intersection $\cap_i U_i$ is dense.

   Let $U \subset \mathbb{R}_\ell$ be a dense open subset. We define $U' = \cup \{(a, b) \mid (a, b) \subset U\}$. Show that $U'$ is dense in $\mathbb{R}$. Given $a < b$, we have $(a, b) \cap U \neq \emptyset$ since $U$ is dense in $\mathbb{R}_\ell$. Let $z \in (a, b) \cap U$. Then there is $d$ with $b > d > z$ such that $[z, d) \subset U$. Then $(z, d) \subset U'$. Hence $(a, b) \cap U' \neq \emptyset$. Thus, $U'$ is dense in $\mathbb{R}$.

   If $U_i$ is a sequence of dense open sets in $\mathbb{R}_\ell$, then $U_i'$ is a sequence of dense open sets in $\mathbb{R}$. Since $\mathbb{R}$ is baire, the set $Z = \cap_i U_i'$ is dense in $\mathbb{R}$. For every $a < b$ the intersection $Z \cap [a, b) \neq \emptyset$, since $Z \cap (a, b) \neq \emptyset$. Thus, $Z$ is dense in $\mathbb{R}_\ell$. Therefore, $\cap_i U_i \supset Z$ is dense in $\mathbb{R}_\ell$.

   (b) No. Every contractible space is path connected. Note that $\mathbb{R}_\ell$ is not connected.

2. Is $A$ a retract of $\mathbb{R}^3$ where
   (a) $A$ is a two point set?
   (b) $A$ is the $x$-axis?
   (c) $A = \{(x, y, z) \mid (x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 2 \& x + y + z = 1\}$
   (d) $A$ is the knotted $x$-axis?
   SOLUTION:
   (a) No, since the image under any continuous map of a connected set must be connected.

   (b) Yes, $r : \mathbb{R}^3 \rightarrow \mathbb{R} \times 0 \times, r(x, yz) = x$, is a retraction.

   (c) No, since $A$ is the intersection of a sphere and a non-tangent plane, $A$ is a circle. The inclusion $j : A \rightarrow \mathbb{R}^3$ is nullhomotopic. Hence $j_*$ is zero homomorphism of the fundamental groups. If there is a retraction, then $j_*$ is a monomorphism. Since $\pi_1(A) \neq 0$, this brings a contradiction.

   (d) Yes. Fix a homeomorphism $h : A \rightarrow \mathbb{R}$. By the Tietze extension theorem there is an extension $\tilde{h} : \mathbb{R}^3 \mathbb{R}$. Define $r : \mathbb{R}^3 \rightarrow A$ as $r = h^{-1} \circ \tilde{h}$. Then $r$ is a retraction: $r(a) = h^{-1}(h(a)) = h^{-1}(h(a)) = a$ for every $a \in A$. 

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3. Show that the Stone-Čech compactification $\beta X$ of $X$ is connected if and only if $X$ is connected.

SOLUTION. If $X$ is connected then $\beta X$ is connected as the closure of connected set.

If $X$ is not connected, then there is a separation $X = A \bigsqcup B$. Then the function $f : X \to \mathbb{R}$ defined as $f(A) = 0$ and $f(B) = 1$ is continuous and bounded. Hence $f$ extends to a continuous function $\bar{f} : \beta X \to \{0, 1\}$. Therefore $\beta X$ is not connected.

4. Let $X = S^1 \times I$ and let $x_0 = (1, 0)$ where $1 \in S^1 \subset \mathbb{C}$. Consider paths $f, g, h : I \to X$ defined by the formulas: $f(s) = (e^{2\pi is}, 1)$, $p(s) = (1, s)$, and $h(s) = (e^{2\pi is}, s)$. Do the loops $p * f * \bar{p}$ and $h * f * \bar{h}$ define the same element of the fundamental group $\pi_1(X, x_0)$?

SOLUTION. Yes. Let $r : S^1 \times I \to S^1$ be the projection. We identify $S^1$ with $S^1 \times \{0\}$. Since $r$ is a deformation retraction onto $S^1 \times \{0\}$, $r^*$ is an isomorphism. Let $\phi(s) = e^{2\pi is}$. Then $r^*([p * f * \bar{p}]) = [c_1 * \phi * c_1] = [\phi]$ and $r^*([h * f * \bar{h}]) = [\phi * \phi * \bar{\phi}]) = [\phi]$. Therefore, $[p * f * \bar{p}] = [h * f * \bar{h}]$.

5. Show that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

SOLUTION. We note $\mathbb{R}^2 \setminus pt$ is homotopy equivalent to $S^1$ and $\mathbb{R}^3 \setminus pt$ is homotopy equivalent to $S^2$. Since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(S^2) = 0$, the spaces $\mathbb{R}^2 \setminus pt$ and $\mathbb{R}^3 \setminus pt$ have nonisomorphic fundamental groups and, hence, cannot be homeomorphic. As a corollary we obtain that $\mathbb{R}^2$ and $\mathbb{R}^3$ cannot be homeomorphic.

6. Let $X_{\geq 0}$, $Y_{\geq 0}$, and $Z_{\geq 0}$ denote the nonnegative parts of the $x$-axis, the $y$-axis, and the $z$-axis in $\mathbb{R}^3$ respectively. Let $A = \mathbb{R}^3 \setminus (X_{\geq 0} \cup Y_{\geq 0} \cup Z_{\geq 0})$.

(a) Is $\pi_1(A)$ infinite?

(b) Is $\pi_1(A)$ abelian?

SOLUTION. Consider the standard deformation retraction of $\mathbb{R}^3 \setminus 0$ onto the unit sphere defined by the straight line homotopy $H(x; t) = (1-t)x + t\frac{x}{||x||}$. This deformation moves each point $x$ along the ray issued from the origin through $x$. Hence the restriction of $H$ onto $A \times I$ is a deformation retraction. Note that the image of $A$ under the restriction is the 2-sphere with three points removed: $S^2 \setminus \{(1; 0; 0); (0; 1; 0); (0; 0; 1)\}$. A 2-sphere with three points deleted is homeomorphic to $\mathbb{R}^2$ with two points deleted. This space is homotopy equivalent to the wedge of two circles which has the fundamental group $F_2$. Thus it is both infinite and nonabelian.
7. Show that if \( g : S^2 \to S^2 \) is continuous and \( g(x) \neq g(-x) \) for all \( x \), then \( g \) is surjective.

SOLUTION: Assume that there is \( a \in S^2 \setminus g(S^2) \). We know that there is a homeomorphism \( h : S^2 \to \mathbb{R}^2 \). Consider the map \( h \circ g : S^2 \to \mathbb{R}^2 \). By the Borsuk-Ulam theorem there is \( x \in S^2 \) such that \( h(g(x)) = h(g(-x)) \). Since \( h \) is injective, we obtain \( g(x) = g(-x) \). We have a contradiction.

8. By the definition the triod (or the letter T space) is a space homeomorphic to \( X = [0, 1] \times \{0\} \cup \{0\} \times [-1, 1] \subset \mathbb{R}^2 \). Show that every continuous map of the triod to itself has a fixed point.

SOLUTION: Clearly, there is a retraction \( r : B \to T \) of the closed 2-disk onto the triod. Let \( f : T \to T \) be a continuous map. Then by the Brouwer Theorem \( f \circ r : B \to B \) has a fixed point \( x \in B \). Note that \( x \in f(r(B)) = T \). Hence \( x \) is a fixed point for \( f \).