1. Suppose that \( X \) is obtained from a lens space \( L_p(\ell_1, \ldots, \ell_n) \) of dimension \( 2n - 1 \) by deleting a point. Compute \( \pi_i(X) \) for \( i < 2n - 1 \).

SOLUTION. The following is assumed to be known about lens spaces \( L_p = L_p(\ell_1, \ldots, \ell_n) \): \( n > 1 \), there is a CW complex structure \( L_p = e^0 \cup e^1 \cup \cdots \cup e^{2n-1} \) with one cell in each dimension, \( \pi_1(L_p) = \mathbb{Z}_p \), the universal covering space is \( S^{2n-1} \). Thus, the universal covering map \( u : S^{2n-1} \to L_p(\ell_1, \ldots, \ell_n) \) is \( p \)-to-1, i.e. the preimage \( u^{-1}(x_0) \) consists of \( p \) points. The restriction of \( u \) over \( L_p \setminus \{x_0\} \) defines a covering map \( u' : S^{2n-1} \setminus u^{-1}(x_0) \to L_p \setminus \{x_0\} \). Then \( \pi_i(L_p \setminus \{x_0\}) = \pi_i(S^{2n-1} \setminus u^{-1}(x_0)) \) for \( i > 1 \). By induction on \( m \) and using suspension one can prove that the \( m \)-sphere \( S^m \) with \( p \) points deleted is homotopy equivalent to the wedge \( \vee_{p-1} S^{m-1} \) of \( m - 1 \)-spheres. Thus,

\[
\pi_{2n-2}(L_p \setminus \{x_0\}) = \pi_{2n-2}(\vee_{p-1} S^{2n-2}) = \mathbb{Z}^{p-1}
\]

and

\[
\pi_i(L_p \setminus \{x_0\}) = \pi_i(\vee_{p-1} S^{2n-2}) = 0
\]

for \( 1 < i < 2n - 2 \). We may assume that \( x_0 \in e^{2n-1} \). Then \( L_p \setminus \{x_0\} \) can be deformed to the \((2n-2)\)-skeleton of \( L_p \). Thus, since \( n > 1 \),

\[
\pi_1(L_p \setminus \{x_0\}) = \pi_1(L_p^{2n}) = \pi_1(L_p) = \mathbb{Z}_p.
\]

An alternative way to prove this is the homotopy exact sequence of the fibration \( u \).

2. Show that the suspension of an acyclic CW complex is contractible.

SOLUTION. It suffices to show that the suspension \( SX \) of an acyclic complex has trivial homotopy groups. Then the constant map \( SX \to pt \) would be a weak homotopy equivalence, and, hence, a homotopy equivalence when \( X \in CW \). Since \( SX \) is also acyclic, in view of Hurewicz theorem it suffices to show that \( SX \) is simply connected. Note that the acyclicity of \( X \) implies that \( X \) is connected. Then we can apply the van Kampen theorem to obtain that \( \pi_1(SX) = \pi_1(CX \cup CX) = 0 \).

3. Show that a map \( f : X \to Y \) between simply connected CW complexes is a homotopy equivalence if and only if the mapping cone \( C_f \) is contractible.

SOLUTION. Assume that \( C_f \) is contractible. Since \( X \) and \( Y \) are simply connected, in view of the homology version of Whitehead theorem, it suffice to show that \( f \) induces isomorphism of homology groups.
This fact follows from the homology exact sequence of the pair \((M_f, X)\) and the fact that \(C_f = M_f/X\) where \(M_f\) is the mapping cylinder.

Assume that \(f\) is a homotopy equivalence. Then the homotopy exact sequence of the pair \((M_f, X)\) implies that \((M_f, X)\) is \(n\)-connected for all \(n\). Then by the Homotopy Excision \(\pi_n(M_f, X) \to \pi_n(M_f/X)\) is an isomorphism for all \(n\). Hence \(\pi_n(C_f) = 0\) for all \(n\). Therefore \(C_f\) is weakly homotopy equivalent to a point. Since \(C_f\) is a CW complex, \(C_f\) is homotopy equivalent to a point, i.e. \(C_f\) is contractible.

4. Show that a map \(f : X \to Y\) between connected CW complexes is a homotopy equivalence if it induces an isomorphism on \(\pi_1\) and if a lift \(\tilde{f} : \tilde{X} \to \tilde{Y}\) to the universal covers induces isomorphism of homology groups.

SOLUTION. Note that by the Whitehead theorem \(\tilde{f} : \tilde{X} \to \tilde{Y}\) induces isomorphism of the homotopy groups. Therefore, \(f\) induces isomorphism of the \(n\)-homotopy groups for \(n > 1\). By Whitehead theorem \(f\) is a homotopy equivalence.

5. If \(X\) is a \(K(G, 1)\) CW-complex, show that \(\pi_n(X^n)\) is free abelian for \(n \geq 2\). Here \(X^n\) is the \(n\)-skeleton of \(X\).

SOLUTION. Note that the universal cover \(\tilde{X}\) contractible. Hence \(\pi_i(\tilde{X}^n) = 0\) for \(i < n\). By Hurewicz theorem \(\pi_n(\tilde{X}) = H_n(\tilde{X})\). By the homology exact sequence of the pair \((\tilde{X}^n, \tilde{X}^{n-1})\) and the fact that \(H_n(\tilde{X}^{n-1}) = 0\) it follows that there is an injective homomorphism

\[ H_n(\tilde{X}^n) \to H_n(\tilde{X}^n, \tilde{X}^{n-1}) = H_n((\tilde{X}^n/\tilde{X}^{n-1}) = H_n(\vee S^n) = \oplus \mathbb{Z}. \]

Therefore, \(H_n(\tilde{X}^n)\) is free as a subgroup of a free abelian group. Note that \(\pi_n(X^n) = \pi_n(\tilde{X}^n) = H_n(\tilde{X}^n)\).

6. Show that if \(S^k \to S^m \to S^n\) is a fiber bundle, then \(k = n - 1\) and \(m = 2n - 1\).

SOLUTION. We assume that \(m > n\). From the homotopy exact sequence of fibration \(p : S^m \to S^n\) it follows that \(\mathbb{Z} = \pi_n(S^n) = \pi_{n-1}(S^k)\) and \(\pi_i(S^k) = 0\) for \(i < n - 1\). This implies that \(k = n - 1\). Then \(m = 2n - 1\) by dimensional reason.

If \(m = n\), then \(p\) is a covering map with a discrete fiber \(F\). Thus, \(F = S^0\). A covering map exists only if \(n = 1\).

7. Let \(E = \mathbb{R}^2 \setminus A\) and \(p : E \to \mathbb{R}\) is the projection \((x, y) \mapsto x\). Is \(p\) a fiber bundle if

(a) \(A = (0, 1) \times \mathbb{R}_+\) ?
(b) \( A = [0, 1] \times \mathbb{R}_+ \)?

(c) \( A = \{0\} \times \mathbb{R}_+ \)?

where \( \mathbb{R}_+ = [0, \infty) \).

**SOLUTION.** (a) No. Every fiber bundle is an open map. The map \( p \) is not open since the image of open set \( E \cap (-1, 1) \times (1, 2) \) equals \( (-1, 0] \) which is not open in \( \mathbb{R} \).

(c) Yes. There is a fiber preserving homeomorphism \( h : (-\infty, 0) \times \mathbb{R} \to E \) defined as \( h(a, x) = (x, -\frac{1}{a}x^2 + a) \). Hence \( p \) is a trivial fiber bundle.

(b) Yes. This \( p \) is the pull-back of \( p \) from (c) under the map \( f : \mathbb{R} \to \mathbb{R} \) that collapses \([0, 1]\) to a point. Therefore, it is a fiber bundle.

8. Show that a map \( p : E \to B \) is a Hurewicz fibration if and only if the map \( \pi : E^I \to E_p, \pi(\gamma) = (\gamma(0), p\gamma) \) has a section.

**SOLUTION.** Suppose that there is a section \( s : E_p \to E^I \) of \( \pi \). Given a homotopy \( H : X \times I \to B \) with the initial lift \( h : X \times 0 \to E \) we define a covering homotopy \( \tilde{H} : X \times I \to E \) as follows. The maps \( H \gamma \) and \( h \) define a map \( g : X \to E_p \) as \( g(x) = (h(x, 0), H|_{x \times I}) \). Then \( \tilde{H}(x, t) = (sg(x))(t) \). Indeed, \( \pi sg(X) = (sg(x)(0), psg(x)) = (h(x, 0), H|_{x \times I}) \) and hence \( \tilde{H} \) is a covering homotopy for \( H \) with initial condition \( h \).

If \( p \) is a fibration then the homotopy \( H : E_p \times I \to B \) defined as \( H((x, \gamma), t) = \gamma(t) \) with the initial lift \( h : E_p \times 0 \to E \) defined as \( h(x, \gamma, 0) = x \) admits a covering homotopy \( \bar{H} : E_p \times I \to E \). This map defines a map \( s : E_p \to E^I \) which is the required section.

9. Let \( G \) be a topological group. Show that a principal fiber \( G \)-bundle \( G \to E \to B \) is trivial if and only if it has a section, a map \( s : B \to E \) with \( ps = 1 \).

**SOLUTION.** Suppose that \( p : G \times B = E \to B \) is a trivial principal \( G \)-bundle. Then every map \( f : B \to G \) defines a section \( s : B \to E \). Every section of \( p \) defines a new trivialization \( h : G \times B \to G \times B \) by the formula \( h(g, b) = (gf(b), b) \).

Now assume that a principal fiber \( G \)-bundle \( p : E \to B \) has a section \( s : B \to E \). We define a trivialization map \( h : G \times B \to E \) as \( h(g, b) = g(s(b)) \) where \( g : E \to E \) is the action by an element \( g \in G \). Clearly, \( h \) is a continuous bijection. The inverse \( h^{-1} \) is continuous, since its restriction to \( p^{-1}(U) \) is continuous for any open sets \( U \subset B \) over which \( p \) is a trivial.