FINAL SOLUTIONS

Spring 2024

1. Show that every map $f : S^{k+l} \to S^k \times S^l$, $k, l > 0$, induces the trivial homomorphism of reduced cohomology groups

$$f^* : \tilde{H}^*(S^k \times S^l) \to \tilde{H}^*(S^{k+l}).$$

SOLUTION: The cohomology ring $H^*(S^m) = \mathbb{Z}[\alpha]/(\alpha^2)$ with $|\alpha| = m$. Since the groups $H^r(S^m)$ are free for all r and m, the cohomology ring $H^*(S^k \times S^l)$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta^2)$. Since $k, l < k + l$, we obtain $H^k(S^{k+l}) = 0 = H^l(S^{k+l})$. Hence $f^*(\alpha) = 0 = f^*(\beta)$. Thus, f^* is zero homomorphism in dimensions > 0.

2. Show that there is no map $f : \mathbb{C}P^m \to \mathbb{C}P^n$ with $m > n$ inducing a nontrivial homomorphism $f^* : H^2(\mathbb{C}P^n) \to H^2(\mathbb{C}P^m)$.

SOLUTION: Note that $H^*(\mathbb{C}P^r) = \mathbb{Z}[\alpha_r]/(\alpha_r^{r+1})$ with $|\alpha_r| = 2$. If $f^*(\alpha_n) \neq 0$, then $f^*(\alpha_n) = k\alpha_m$ with $k \neq 0$. Then we would have a contradiction:

$$0 = f^*(\alpha_n^{n+1}) = f^*(\alpha_n)^{n+1} = k^{n+1}\alpha_m^{n+1} \neq 0.$$

3. Show that RP^{2n+1} is not homotopy equivalent to $RP^{2n} \vee S^{2n+1}$

SOLUTION: The cohomology rings taken with $\mathbb{Z}_2$ coefficients

$$H^*(RP^{2n+1}) = \mathbb{Z}_2[\alpha]/(\alpha^{2n+2})$$

and

$$H^*(RP^{2n} \vee S^{2n+1}) = H^*(RP^{2n}) \oplus H^*(S^{2n+1}) = \mathbb{Z}_2[\beta]/(\beta^{2n+1}) \oplus \mathbb{Z}_2[\gamma]/(\gamma^2)$$

are not isomorphic since the first contains an element α whose $(2n+1)$st power α^{2n+1} is nonzero and in the second $x^{2n+1} = 0$ for all elements.

4. Let $U \subset \mathbb{R}^n$ be an open set. Show that the groups $H_i(U)$ are countable for all i.

SOLUTION: Consider the partition of $\mathbb{R}$ into intervals $[\epsilon n, \epsilon(n+1)]$, $n \in \mathbb{Z}$ of length ϵ. It defines a CW complex structure on $\mathbb{R}$. An ϵ cubical n-complex is a subcomplex of $\mathbb{R}^n$ with respect to the product CW structure. For an open set $U \subset \mathbb{R}^n$ we denote by C_n the union of $\frac{1}{2^n}$-cubes that are contained in U. Note that C_n is a finite CW complex, $C_{n+1} \subset C_n$, $U = \cup C_n$, and every compact set $X \subset U$ lies in some C_n.

Then $H_i(U) = \lim_\to H_i(C_n)$. Since $H_i(C_n)$ are countable, $\oplus_n H_i(C_n)$ is countable and, hence, $\lim_\to H_i(C_n)$ is countable.

5. Show that a compact manifold does not retract onto its boundary.
SOLUTION: Let M be a compact n-manifold with boundary. We consider the homology exact sequence of the pair $(M, \partial M)$ for $\mathbb{Z}_2$ coefficients:

$$H_n(M;\mathbb{Z}_2) \to H_n(M,\partial M;\mathbb{Z}_2) \stackrel{\partial}{\to} H_{n-1}(\partial M;\mathbb{Z}_2) \stackrel{i_*}{\to} H_{n-1}(M;\mathbb{Z}_2) \to .$$

If ∂M is a retract of M, then i_* is injective. Since M is $\mathbb{Z}_2$ orientable, $H_n(M,\partial M;\mathbb{Z}_2) \neq 0$. It was proven in class that every noncompact n-manifold has trivial n-homology for all coefficients. Hence $H_n(M;\mathbb{Z}_2) \cong H_n(M \setminus \partial M;\mathbb{Z}_2) = 0$. Therefore, ∂ is nonzero homomorphism and i_* cannot be injective. Contradiction.

6. Show that $H_c^{n+1}(X \times \mathbb{R}) = H_c^n(X)$ for all n.
SOLUTION: We use the fact (proven in class) that the cross product with the generator of $H^1(I, \partial I)$ induces an isomorphism $H^n(Y) \to H^{n+1}(Y \times I, Y \times \partial I)$. In the case of the relative cohomology, this cross product defines an isomorphism $H^n(Y, A) \to H^{n+1}(Y \times I, Y \times \partial I \cup A \times I)$. Then we obtain a chain of isomorphisms

$$H_c^{n+1}(X \times \mathbb{R}) = \lim_{\to\, C \subset_{comp} X, m \in \mathbb{N}} H^{n+1}(X \times \mathbb{R} \mid C \times [-m,m]) =$$

$$= \lim_{\to} H^{n+1}(X \times \mathbb{R}, X \times \mathbb{R} \setminus (C \times [-m,m]) =$$

$$(h.e.) = \lim_{\to} H^{n+1}(X \times \mathbb{R}, X \times \mathbb{R} \setminus (C \times (-m,m)) =$$

$$(excision) = \lim_{\to} H^{n+1}(X \times [-m,m], X \times [-m,m] \setminus (C \times (-m,m)) =$$

$$= \lim_{\to} H^{n+1}(X \times [-m,m], X \times \partial[-m,m] \cup (X \setminus C \times [-m,m]) =$$

$$(cross\ product\ isomorphism) = \lim_{\to} H^n(X, X \setminus C) = H_c^n(X).$$

7. Show that the tensor product commutes with direct limits:

$$(\lim_{\to} G_\alpha) \otimes A = \lim_{\to}(G_\alpha \otimes A).$$

SOLUTION: Note that the statement holds true when $A = \oplus \mathbb{Z}$. Let $F_1 \to F_0 \to A \to 0$ be a free resolution of A. Thus $F_i = \oplus \mathbb{Z}$, $i = 0, 1$, and the sequence is exact. The sequence

$$(\lim_{\to} G_\alpha) \otimes F_1 \to (\lim_{\to} G_\alpha) \otimes F_2 \to (\lim_{\to} G_\alpha) \otimes A \to 0$$

is exact since the tensor product is a right exact functor. For each $\alpha \leq \beta$ there is a commutative diagram:

$$\begin{array}{ccccccc}
(\lim_{\to} G_\alpha) \otimes F_1 & \longrightarrow & (\lim_{\to} G_\alpha) \otimes F_2 & \longrightarrow & (\lim_{\to} G_\alpha) \otimes A & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
G_\beta \otimes F_1 & \longrightarrow & G_\beta \otimes F_0 & \longrightarrow & G_\beta \otimes A & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
G_\alpha \otimes F_1 & \longrightarrow & G_\alpha \otimes F_0 & \longrightarrow & G_\alpha \otimes A & \longrightarrow & 0.
\end{array}$$

Hence there is the limit diagram which is also commutative:

$$\begin{array}{ccccccc}
(\lim_{\to} G_\alpha) \otimes F_1 & \longrightarrow & (\lim_{\to} G_\alpha) \otimes F_2 & \longrightarrow & (\lim_{\to} G_\alpha) \otimes A & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\lim_{\to}(G_\alpha \otimes F_1) & \longrightarrow & \lim_{\to}(G_\alpha \otimes F_0) & \longrightarrow & \lim_{\to}(G_\alpha \otimes A) & \longrightarrow & 0.
\end{array}$$

The bottom row is exact as the direct limit of exact sequences. Two left vertical arrows are isomorphism since F_i are free. By the Five Lemma we obtain the result.

8. Show that for orientable surfaces M_g and M_h of genus g and h with $g > h$

(a) there is no map $f : M_h \to M_g$ of degree one;

(b) there is no retraction of M_{g+h} onto $M_g \setminus IntD^2$ where $M_{g+h} = M_g \# M_h$ be the connected sum, i.e.,

$$M_{g+h} = (M_g \setminus IntD^2) \cup_{\partial D^2} (M_h \setminus IntD^2).$$

SOLUTION: (a) We show that a degree one map $f : M \to N$ between closed orientable n-manifolds induces an epimorphism of homology groups. For any $a \in H_i(N)$ by the Poincare duality there exists a cohomology class $\alpha \in H^{n-i}(N)$ dual to a, i.e., $[N] \cap \alpha = a$. The Naturality of the cap product fomula

$$f_*(\sigma \cap f^*\phi) = f_*(\sigma) \cap \phi$$

implies that

$$f_*([M] \cap f^*(\alpha)) = f_*([M]) \cap \alpha = [N] \cap \alpha = a.$$

Note that $rank(H_1(M_h)) < rank(H_1(M_g)$ when $g > h$. Hence f_* cannot be surjective. Thus, there is no degree one map $f : M_h \to M_g$ for $g > h$.

(b) Let $r : M_{g+h} \to M_g \setminus IntD^2$ be a retraction. We define a map $f : M_h \to M_g$ as follows. We set $f = r$ on $M_h \setminus IntD^2$ and $f = id$ on D^2. Since r is a retraction, these two functions agree on ∂D^2 and f is continuous by the Pasting Lemma. Note that $deg f = 1$ since the local degree of id is 1 and we can compute the degree of f using local degrees of the preimage $f^{-1}(c)$ where c is the center of D^2. This contradicts to (a).