1. Let $q : \Delta^2 \to X$ be the quotient map obtained by preserving the ordering identification of the edges $[v_0, v_1]$ and $[v_1, v_2]$ in the simplex $\Delta^2 = [v_0, v_1, v_2]$.

(a) Compute the homology groups $H_*(X)$;

SOLUTION. Let $[v_0, v_1] = [v_1, v_2] = a$ and $[v_0, v_2] = b$. The $\Delta$-chain complex of $X$ is

$$0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}\langle a, b \rangle \xrightarrow{0} \mathbb{Z} \to 0$$

with $\partial(1) = a - b + a$. Then $\text{ker}(\partial) = 0$ and, hence $H_2(X) = 0$. Clearly, $H_0(X) = \mathbb{Z}$. Since $\mathbb{Z}\langle a, b \rangle = \mathbb{Z}\langle 2a - b, b - a \rangle$, We obtain $H_1(X) = \mathbb{Z}\langle a, b \rangle / \text{im}\partial = \mathbb{Z}$.

(b) Compute the relative homology groups $H_*(X, A)$ where $A = q([v_0, v_2])$.

SOLUTION. The relative chain complex is

$$0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}\langle a \rangle \to 0$$

with $\partial(1) = 2a$. Thus, $H_0(X, A) = 0$, $H_1(X, A) = \mathbb{Z}_2$, $H_2(X, A) = 0$.

2. Compute the fundamental group and the homology groups of the Klein bottle.

SOLUTION. The Klein bottle $K$ is obtained from the square by identification of opposite sides for one pair with the same direction and the other with the opposite. Thus we have one vertex $v$, two 1-cells $a$ and $b$ and one 2-cell $e$. The fundamental group has a presentation

$$\langle a, b \mid abab^{-1} = 1 \rangle.$$ 

Note that there are many equivalent presentations.

The cellular chain complex is

$$0 \to \mathbb{Z} < e > \xrightarrow{d_2} \mathbb{Z} < a, b > \xrightarrow{d_1} \mathbb{Z} < v > \to 0$$

with $d_1 = 0$ and $d_2(e) = \lambda_1a + \lambda_2b$ where $\lambda_1$ and $\lambda_2$ the degree of the attaching map followed by projection on the circles $a \cup v$ and $b \cup v$ respectively. Assume that the sides of the square labeled by $b$ are identified with the opposite direction. Then $\lambda_1 = 0$ and $\lambda_2 = 2$. Then $d_2$ is injective and hence, $H_2(K) = 0$. $H_0(K) = \mathbb{Z}$ since $K$ is path connected. Note that

$$H_1(K) = (\mathbb{Z} \oplus \mathbb{Z})/(0 \oplus 2\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2.$$
3. Let $X$ be the mapping cone of a map $p : S^1 \to S^1$ of degree $p$.
   (a) Compute the groups $H_n(X)$;
   SOLUTION. The space has the following CW complex structure:
   $X = e^0 \cup e^1 \cup e^2$. The cellular chain complex is
   \[ 0 \to \mathbb{Z}\langle e^2 \rangle \xrightarrow{p} \mathbb{Z}\langle e^1 \rangle \to \mathbb{Z}\langle e^0 \rangle \to 0. \]
   Then $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_p$, $H_2(X) = 0$.

   (b) Compute the groups $H_n(\tilde{X})$ where $\tilde{X}$ is the universal cover of $X$.
   SOLUTION. We claim that the universal cover $\tilde{X}$ is $p$ copies of 2-disks with the same boundary with the $\mathbb{Z}_p$-action that the cyclic permutation on 2-disc with the rotation on $2\pi/p$. Indeed, it is easy to see that the action is free and the orbit space is $X$. Since collapsing of a contractible subcomplex is a homotopy equivalence, $\tilde{X}$ is homotopy equivalent to $\tilde{X}/D^2 = \vee_{p-1} S^2$ where $D^2$ is on of the 2-disks. Hence, $H_*(\tilde{X}) = H_*(\vee_{p-1} S^2)$. Thus, $H_0(\tilde{X}) = \mathbb{Z}$, $H_1(\tilde{X}) = 0$, and $H_2(\tilde{X}) = \oplus_{p-1} \mathbb{Z}$.

4. Compute the fundamental group of the quotient space of an annulus obtained by identifying antipodal points on the outer circle and identifying points on the inner circle which are $2\pi/3$-apart.
   SOLUTION. By collapsing an interval joining the inner and the outer circles we obtain a homotopy equivalent complex $X$ with the CW-complex structure $X = e^0 \cup a \cup b \cup e^2$ where $a$ and $b$ are 1-cells and the attaching map of the 2-cell is defined by the word $a^2b^{-3}$. Thus we have $\pi_1(X) = \langle a, b \mid a^2b^{-3} = 1 \rangle$.

5. Find all covering spaces of $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$.
   SOLUTION. The fundamental group $\pi_1(X) = \mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group which has a presentation
   \[ D_\infty = \langle a, x \mid x^2 = 1, xax = a^{-1} \rangle. \]
   We call the projective planes in the wedge *orange* and *blue*. The universal cover $\tilde{X}$ is the string of spheres of alternating colors. The generator $a$ acts on $\tilde{X}$ by translation by two: Each orange sphere goes to the next orange sphere. The generator $x$ acts by the central symmetry in the center of one of the spares. Let $H$ be a proper subgroup of $D_\infty$.
   If $H$ is contained in $\langle a \rangle$, then it is generated by $\langle a^n \rangle$ for some $n$ and has index $2n$. Then the covering space $X_H$ is the circular string of $2n$ spheres of alternating color which obtained from $\tilde{X}$ from the natural action of $H$ on it.
If \( H \cap \langle a \rangle = \emptyset \), Then \( H \) has order 2. So it acts by the central symmetry on \( \bar{X} \). The orbits space \( X_H \) is a half-line string of spheres that starts with the projective plane. There are two cases here defined by the color of the end projective plane.

If \( H \cap \langle a \rangle = \langle a^n \rangle \) for some \( n \) and \( H \) contains some \( xa^k \) then up to conjugacy we may assume that \( k = 0 \) or \( k = 1 \). Then the index of \( H \) is \( n \). Then \( X_H = \bar{X}/H \) is the string of \((n-1)\)-spheres with the projective planes at the ends. If \( n \) is even, both projective planes are of the same color. If \( n \) is odd, they are of different colors.

6. Compute the Euler characteristic \( \chi(T^n) \) of the \( n \)-dimensional torus and the group \( H_n(T^n) \).

**SOLUTION.** Since \( \chi(S^1) = 0 \) and \( \chi(X \times Y) = \chi(X)\chi(Y) \), we obtain \( \chi(T^n) = 0 \).

**Claim:** \( H_n(T^n) = \mathbb{Z} \).

**1st proof:** In the CW complex structure that comes as the product from the minimal CW structure on \( S^1 \), the torus \( T^n \) has one \( n \)-cell. It suffices to show that the attaching map of the top cell has local degree 0 over each \((n-1)\)-cell. We can see the attaching map \( \phi \) as the map identifying opposite faces in the \( n \)-cube. Each pair of opposite faces represents a \((n-1)\)-cell in \( T^n \). Then for every point \( y \) in the (open) \((n-1)\) cell the preimage \( \phi^{-1}(y) = \{x_-, x_+\} \) consists of two points which are the mirror image of each other with respect to the hyperplane through the origin which is parallel to our opposite faces. Since the reflection has degree -1, the result follows.

**2nd proof:** We prove by induction on \( n \) that \( H_n(T^n) = \mathbb{Z} \). It’s a true statement for \( n = 1 \). Let \( s \in S^1 \) be a fixed point and let \( n > 1 \). There is a retraction of \( T^n = T^{n-1} \times S^1 \) onto \( T^{n-1} \times s \). Therefore,

\[
\tilde{H}_i(T^n) = \tilde{H}_i(T^{n-1}) \oplus H_i(T^n, T^{n-1} \times s).
\]

Note that \( H_i(T^n, T^{n-1} \times s) = \tilde{H}_i(T^n/(T^{n-1} \times s)) \). CLAIM: The space \( T^n/(T^{n-1} \times s) \) is homotopy equivalent to the wedge \( ST^{n-1} \vee S^1 \) of the suspension \( ST^{n-1} \) and a circle. PROOF: We may replace the circle in the wedge by an interval \( I \) attached to the vertices of the suspension \( ST^{n-1} \). Collapsing \( I \) to a point gives the space \( T^n/(T^{n-1} \times s) \).

Thus,

\[
H_n(T^n) = H_n(T^{n-1}) \oplus H_n(ST^{n-1}) \oplus H_n(S^1).
\]

Note that \( H_n(T^{n-1}) = 0 = H_n(S^1) \) by dimensional reasons. Hence,

\[
H_n(T^n) = H_n(ST^{n-1}) = H_{n-1}(T^{n-1}) = \mathbb{Z}.
\]
7. Compute the homology groups of $K \times S^1$ where $K$ is the Klein bottle.

**SOLUTION.** Let $s \in S^1$. Note that $K \times S^1/K \times s$ is homotopy equivalent to $SK \vee S^1$. From the exact sequence of the pair $(K \times S^1, K \times s)$ and the existence of a retraction $K \times S^1 \to K \times s$ we obtain

$$\tilde{H}_i(K \times S^1) = \tilde{H}_i(K) \oplus H_i(K \times S^1, K \times s).$$

Thus,

$$\tilde{H}_i(K \times S^1) = \tilde{H}_i(K) \oplus \tilde{H}_{i-1}(K) \oplus \tilde{H}_i(S^1) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & i = 2 \\ 0 & \text{otherwise} \end{cases}.$$  

8. Does there exist a covering space of the surface $M_3$ of genus 3 with

(a) the deck transformation group $\mathbb{Z} \times \mathbb{Z}$?

**SOLUTION.** Yes, since there is an epimorphism $\phi : \pi_1(M_3) \to \mathbb{Z} \times \mathbb{Z}$. For example, the map $q : M_3 = T \# T \# T \to T$ that collapses $T \# T$ to a point induces such. The covering space $X$ that corresponds to the subgroup $H = \ker(\phi) \subset \pi_1(M_3)$ is normal (since) $H$ is normal and $\pi_1(M_3)/H = \mathbb{Z} \times \mathbb{Z}$.

(b) the deck transformation group $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$?

**SOLUTION.** Yes, since there is an epimorphism $\phi : \pi_1(M_3) \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. One can define such by going first to the abelianization $\mathbb{Z}^6$ and then by taking the projection.

(c) [Extra Credit] the fundamental group $\mathbb{Z} \times \mathbb{Z}$?

**SOLUTION:** A covering of $M_3$ is a 2-manifold. An open 2-manifold is homotopy equivalent to 1-dimensional complex* and hence has free fundamental group. Thus, it cannot be open. If it closed, then it is torus and the covering is $n$-to-one for $n \in \mathbb{N}$. Then we have a contradiction with Euler characteristic $0 \neq n\chi(M_3) = \chi(T) = 0$.

**Pf** Choose a triangulation of the surface $S$, equipped with the simplicial metric. Choose a maximal one-ended subtree $T$ of the dual 1-skeleton of $S$. The subtree $T$ contains every dual 0-cell, that is, the barycenter of every 2-simplex. Also, $T$ contains dual 1-cells crossing certain 1-simplices. Let $U$ be the union of the open 2-simplices and open 1-simplices that contain a point of $T$. The metric completion of $U$, denoted $U_c$, is a closed disc with one boundary point removed, and
so there is a deformation retraction from $U_c$ onto its boundary $\partial U_c$. Attaching $U_c$ to $S \setminus U$ in the obvious way to form the surface $S$, the deformation retraction $U_c \to \partial U_c$ induces a deformation retraction of $S$ onto $S \setminus U$, which is a subcomplex of the 1-skeleton.