1. (20pts) Solve the differential equation \( y'' + 4y' + 4y = 2x + 6 \).

SOLUTION: The solution is the sum \( y = y_h + y_p \). The auxiliary equation \( m^2 + 4m + 4 = 0 \) can be factored as \( (m + 2)^2 = 0 \). We have one repeated root \( m = -2 \). Thus, the functions \( y_1 = e^{-2x} \) and \( y_2 = xe^{-2x} \) form a fundamental set of solutions of the homogeneous equation \( y'' + 4y' + 4y = (D + 2)^2y = 0 \). Thus, \( y_h = c_1e^{-2x} + c_2xe^{-2} \).

To find \( y_p \) we note that \( D^2 \) annihilates \( 2x + 6 \). Then \( D^2(D + 2)^2y = 0 \) and \( y = y_h + y_p = c_1e^{-2x} + c_2xe^{-2} + c_3 + c_4x \). Since \( y_h = c_1e^{-2x} + c_2xe^{-2} \), we obtain \( y_p = c_3 + c_4x \). Then \( y''_p = c_4 \) and \( y'''_p = 0 \). Plug this all to the equation to obtain \( 0 + 4c_4 + 4(c_3 + c_4x) = 2x + 6 \). Hence, \( 4c_4 = 2 \) and \( 4c_4 + 4c_3 = 6 \). Therefore, \( c_4 = 1/2 \) and \( c_3 = 1 \).

The answer is \( y = c_1e^{-2x} + c_2xe^{-2} + 1 + \frac{1}{2}x \).

2. (20pts) Find a general solution to the equation

\[
(D + 4)(D - 3)(D + 2)^3(D^2 + 4D + 5)^2D^4[y] = 0.
\]

SOLUTION: The roots of the auxiliary equation \((r + 4)(r - 3)(r + 2)^3(r^2 + 4r + 5)^2r^4 = 0\) are

- \( r_1 = -4 \), unrepeated, that gives \( y_1 = e^{-4x} \);
- \( r_2 = 3 \), unrepeated, that gives \( y_2 = e^{3x} \);
- \( r_{3,4,5} = -2 \), repeated of multiplicity 3, that give \( y_3 = e^{-2x} \), \( y_4 = xe^{-2x} \), \( y_5 = x^2e^{-2x} \);
- \( r_{6,7,8,9} = -2 \pm i \), complex conjugate repeated of multiplicity 2, that give \( y_6 = e^{-2x}\cos x \), \( y_7 = e^{-2x}\sin x \), \( y_8 = xe^{-2x}\cos x \), \( y_9 = e^{-2x}\sin x \);
- \( r_{10,11,12,13} = 0 \), repeated of multiplicity 4, which gives \( y_{10} = 1 \), \( y_{11} = x \), \( y_{12} = x^2 \), \( y_{13} = x^3 \).
The answer $y = c_1 e^{-4x} + c_2 e^{3x} + (c_3 + c_4 x + c_5 x^2) e^{-2x} + (c_6 + c_8 x) e^{-2x} \cos x + (c_7 + c_9 x) e^{-2x} \sin x + c_{10} + c_{11} x + c_{12} x^2 + c_{13} x^3$.

3. (20pts) Solve the differential equation $x^2 y'' - 2y = 0$.

SOLUTION: The auxiliary equation is $m^2 - m - 2 = 0$. It can be factorized as $(m - 2)(m + 1) = 0$. Hence $y_1 = x^2$ and $y_2 = x^{-1}$ is a fundamental set of solutions. Thus, $y = c_1 x^2 + c_2 x^{-1}$

4. (20pts) Find a differential operator that annihilates $x^2 e^{-x} \sin(2x)$.

SOLUTION: $((D + 1)^2 + 4)^3$ annihilates $x^2 e^{-x} \sin(2x)$, since $x^2 e^{-x} \sin(2x)$ is a solution to the equation

$$((D + 1)^2 + 4)^3 [x] = 0.$$  

The later holds true since $r = -1 \pm 2i$ is a 3 times repeated root of the auxiliary equation $((r + 1)^2 + 4)^3 = 0$ for $((D + 1)^2 + 4)^3 [x] = 0$.

5. (20pts) Solve the initial value problem

$x' = 4x + y; \quad x(0) = 1,$

$y' = -2x + y; \quad y(0) = 0.$

SOLUTION: The system is equivalent to the following

$(D - 4)[x] - y = 0$

$2x + (D - 1)[y] = 0.$

We multiply the first equation by $D - 1$ and add to the second equation to eliminate $y$:

$(D - 1)(D - 4)[x] + 2x = 0$  $\iff$  $(D^2 - 5D + 6)[x] = 0$  $\iff$

$(D - 2)(D - 3)[x] = 0.$

A general solution to this equation is $x(t) = c_1 e^{3t} + c_2 e^{2t}$. Then $x' = 3c_1 e^{3t} + 2c_2 e^{2t}$. From the first equation we obtain $y = x' - 4x = 3c_1 e^{3t} +$
\[2c_2 e^{2t} - 4(c_1 e^{3t} + c_2 e^{2t}) = -c_1 e^{3t} - 2c_2 e^{2t}.\] The initial conditions give us \(c_1 + c_2 = 1\) and \(-c_1 - 2c_2 = 0\). Thus, 
\(c_1 = -2c_2\) and \(-c_2 = 1\). Therefore, 
\(c_1 = 2\) and \(c_2 = -1\).

The answer is 
\[x = 2e^{3t} - e^{2t}\] 
\[y = -2e^{3t} + 2e^{2t}.\]

6. (Extra Credit 10pts) A mass 64 pounds stretches a spring 0.32 foot.
The mass is released from 8 inches above the equilibrium position with downward velocity 5 ft/s. What are the amplitude and the period of motion? At what times does the mass pass through the equilibrium position heading downward for the second time?

Recall: \(g = 32 \text{ ft/s}^2\).

SOLUTION: We consider coordinate system directed downward with 0 at the equilibrium. By the Second Newton’s law,

\[my'' + k(y + 0.32) = mg.\]

The spring constant \(k\) can be found from the equation \(mg = 0.32k\). Thus, \(64 \times 32 = 0.32k\). So \(k = 6400\). Thus, our equation is

\[64y'' + 6400(y + 0.32) = 64 \times 32\]

which can be simplified to

\[y'' + 100y = 0.\]

A general solution is \(y(t) = c_1 \cos 10t + c_2 \sin 10t.\)

The initial conditions are \(y(0) = -\frac{8}{12} = -\frac{2}{3}\) and \(y'(0) = 5\). They give us 
\(c_1 = -\frac{2}{3}\) and \(10c_2 = 5\). Thus, \(y(t) = -\frac{2}{3} \cos 10t + \frac{1}{2} \sin 10t.\)
The period is \( \frac{2\pi}{\omega} = \frac{\pi}{5} \) where \( \omega = \sqrt{k/m} = \sqrt{6400/64} = 10 \). The amplitude is \( A = \sqrt{\frac{T}{\pi} + \frac{1}{4}} = \sqrt{\frac{25}{36}} = \frac{5}{6} \).

Thus, \( y(t) = \frac{5}{6}\sin(10t + \phi) \) where \( \tan \phi = -\frac{4}{3} \). Let \( t_1 \) be the time when the mass passes through the equilibrium point. Since it was above this point at \( t_0 = 0 \), it will be heading downward at \( t_1 \). Then the second time it will be passing through the equilibrium position heading downward at \( t_2 = t_0 + \frac{\pi}{5} \), since the period is \( \frac{\pi}{5} \). Thus, the time \( t_1 \) is the first root of the equation \( 10t_0 - \tan^{-1}\left(\frac{4}{3}\right) = 0 \). Roughly, \( t_1 = 0.1, \frac{\pi}{5} = 0.6 \), and, hence, \( t_2 = 0.7 \) second.