1. (20pts) Solve the differential equation $y'' + 4y' + 4y = 2x + 6$.

SOLUTION: The solution is the sum $y = y_h + y_p$. The auxiliary equation $m^2 + 4m + 4 = 0$ can be factored as $(m + 2)^2 = 0$. We have one repeated root $m = -2$. Thus, the functions $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ form a fundamental set of solutions of the homogeneous equation $y'' + 4y' + 4y = (D + 2)^2y = 0$. Thus, $y_h = c_1 e^{-2x} + c_2 xe^{-2}$

To find $y_p$, we note that $D^2$ annihilates $2x + 6$. Then $D^2(D + 2)^2y = 0$ and $y = y_h + y_p = c_1 e^{-2x} + c_2 xe^{-2} + c_3 + c_4 x$. Since $y_h = c_1 e^{-2x} + c_2 xe^{-2}$, we obtain $y_p = c_3 + c_4 x$. Then $y'_p = c_4$ and $y''_p = 0$. Plug this all to the equation to obtain $0 + 4c_4 + 4(c_3 + c_4 x) = 2x + 6$. Hence, $4c_4 = 2$ and $4c_4 + 4c_3 = 6$. Therefore, $c_4 = 1/2$ and $c_3 = 1$.

The answer is $y = c_1 e^{-2x} + c_2 xe^{-2} + 1 + \frac{1}{2}x$.

2. (20pts) Find a general solution to the equation $(D + 4)(D - 3)(D + 2)^3(D^2 + 4D + 5)^2D^4[y] = 0$.

SOLUTION: The roots of the auxiliary equation $(r + 4)(r - 3)(r + 2)^3(r^2 + 4r + 5)^2r^4 = 0$ are

$r_1 = -4$, unrepeated, that gives $y_1 = e^{-4x}$;
$r_2 = 3$, unrepeated, that gives $y_2 = e^{3x}$;
$r_{3,4,5} = -2$, repeated of multiplicity 3, that give $y_3 = e^{-2x}$, $y_4 = xe^{-2x}$, $y_5 = x^2e^{-2x}$;
$r_{6,7,8,9} = -2 \pm i$, complex conjugate repeated of multiplicity 2, that give $y_6 = e^{-2x} \cos x$, $y_7 = e^{-2x} \sin x$, $y_8 = xe^{-2x} \cos x$, $y_9 = e^{-2x} \sin x$;
$r_{10,11,12,13} = 0$, repeated of multiplicity 4, which gives $y_{10} = 1$, $y_{11} = x$, $y_{12} = x^2$, $y_{13} = x^3$. 

1
The answer $y = c_1 e^{-4} + c_2 e^{3x} + (c_3 + c_4 x + c_5 x^2) e^{-2x} + (c_6 + c_8 x) e^{-2x} \cos x + (c_7 + c_9 x) e^{-2x} \sin x + c_{10} + c_{11} x + c_{12} x^2 + c_{13} x^3$.

3. (20pts) Solve the differential equation $x^2 y'' - 2y = 0$.
   SOLUTION: The auxiliary equation is $m^2 - m - 2 = 0$. It can be factorized as $(m - 2)(m + 1) = 0$. Hence $y_1 = x^2$ and $y_2 = x^{-1}$ is a fundamental set of solutions. Thus, $y = c_1 x^2 + c_2 x^{-1}$

4. (20pts) Find a differential operator that annihilates $x^2 e^{-x} \sin(2x)$.
   SOLUTION: $((D + 1)^2 + 4)^3$ annihilates $x^2 e^{-x} \sin(2x)$, since $x^2 e^{-x} \sin(2x)$ is a solution to the equation $((D + 1)^2 + 4)^3 [x] = 0$.

The later holds true since $r = -1 \pm 2i$ is a 3 times repeated root of the auxiliary equation $((r + 1)^2 + 4)^3 = 0$ for $((D + 1)^2 + 4)^3 [x] = 0$.

5. (20pts) Solve the initial value problem
   $x' = 4x + y; \quad x(0) = 1,$
   $y' = -2x + y; \quad y(0) = 0.$
   SOLUTION: The system is equivalent to the following
   $(D - 4)[x] - y = 0$
   $2x + (D - 1)[y] = 0$.

   We multiply the first equation by $D - 1$ and add to the second equation to eliminate $y$:
   $(D - 1)(D - 4)[x] + 2x = 0 \iff (D^2 - 5D + 6)[x] = 0 \iff (D - 2)(D - 3)[x] = 0$.

   A general solution to this equation is $x(t) = c_1 e^{3t} + c_2 e^{2t}$. Then $x' = 3c_1 e^{3t} + 2c_2 e^{2t}$. From the first equation we obtain $y = x' - 4x = 3c_1 e^{3t} +$
2c_2 e^{2t} - 4(c_1 e^{3t} + c_2 e^{2t}) = -c_1 e^{3t} - 2c_2 e^{2t}. The initial conditions give us 
\( c_1 + c_2 = 1 \) and \( -c_1 - 2c_2 = 0 \). Thus, \( c_1 = -2c_2 \) and \( -c_2 = 1 \). Therefore, \( c_1 = 2 \) and \( c_2 = -1 \).

The answer is 
\[
x = 2e^{3t} - e^{2t}
\]
\[
y = -2e^{3t} + 2e^{2t}.
\]

6. (Extra Credit 10pts) A mass 64 pounds stretches a spring 0.32 foot. The mass is released from 8 inches above the equilibrium position with downward velocity 5 ft/s. What are the amplitude and the period of motion? At what times does the mass pass through the equilibrium position heading downward for the second time?

Recall: \( g = 32 \text{ ft/s}^2 \).

SOLUTION: We consider coordinate system directed downward with 0 at the equilibrium. By the Second Newton’s law,

\[
my'' + k(y + 0.32) = mg.
\]

The spring constant \( k \) can be found from the equation \( mg = 0.32k \). Thus, \( 64 \times 32 = 0.32k \). So \( k = 6400 \). Thus, our equation is

\[
64y'' + 6400(y + 0.32) = 64 \times 32
\]

which can be simplified to

\[
y'' + 100y = 0.
\]

A general solution is \( y(t) = c_1 \cos 10t + c_2 \sin 10t \).

The initial conditions are \( y(0) = -\frac{8}{12} = -\frac{2}{3} \) and \( y'(0) = 5 \). They give us \( c_1 = -\frac{2}{3} \) and \( 10c_2 = 5 \). Thus, \( y(t) = -\frac{2}{3} \cos 10t + \frac{1}{2} \sin 10t \).
The period is \( \frac{2\pi}{\omega} = \frac{\pi}{5} \) where \( \omega = \sqrt{k/m} = \sqrt{6400/64} = 10 \). The amplitude is \( A = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{5}{6} \).

Thus, \( y(t) = \frac{5}{6} \sin(10t + \phi) \) where \( \tan \phi = -\frac{4}{3} \). Let \( t_1 \) be the time when the mass passes through the equilibrium point. Since it was above this point at \( t_0 = 0 \), it will be heading downward at \( t_1 \). Then the second time it will be passing through the equilibrium position heading downward at \( t_2 = t_0 + \frac{\pi}{5} \), since the period is \( \frac{\pi}{5} \). Thus, the time \( t_1 \) is the first root of the equation \( 10t_0 - \tan^{-1}(\frac{4}{3}) = 0 \). Roughly, \( t_1 = 0.1, \frac{\pi}{5} = 0.6 \), and, hence, \( t_2 = 0.7 \) second.