1. Show that the closed $n$-ball $B = \{ x \in \mathbb{R}^n \mid d(x,0) \leq 1 \}$ has the universal extension property.

SOLUTION: Denote by $\|x\| = d(x,o)$. The set $B$ is a retract of $\mathbb{R}^n$ by means of the map $r : \mathbb{R}^n \to B$ defined as $r(x) = x/\|x\|$ if $\|x\| > 1$ and $r(x) = x$ if $\|x\| \leq 1$. The map $r$ is continuous by the Pasting Lemma: $x/\|x\| = x$ whenever $\|x\| = 1$.

By the Tietze Extension theorem $\mathbb{R}^n$ has the universal extension property: For every normal $X$ and a closed subset $A \subset X$ of a normal space $X$. Let $g : A \to B$ be a continuous map for a closed subset $A \subset X$ of a normal space $X$. Let $\tilde{f} : X \to \mathbb{R}^n$ be an extension of $g$ given by the Tietze Extension theorem. We show that the composition $\tilde{g}r \circ \tilde{f} : X \to B$ is an extension of $g$. Indeed, for $x \in A$ we have $r\tilde{f}(x) = rg(x) = g(x)$, since $r$ is a retraction.

An alternative proof would be to show that the $n$-ball $B$ is homeomorphic to the $n$-cube $I^n$, $I = [0,1]$, which has the universal extension property by the Tietze Extension theorem.

2. Does the sequence $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = x/n$, converge in

(a) the pointwise convergence topology?
(b) the uniform topology?
(c) compact open topology?

SOLUTION: (a) Yes $x/n \to 0$ for any fixed $x$.

(b) No, if it converges the limit must be $0$ but $\rho(x/n,0) = \infty$.

(c) Yes, since for every compact set $C \subset [-M,M]$, $\sup d(x/n,0) \leq M/n \to 0$ as $n \to \infty$.

3. Prove that the set of transcendental numbers $\mathcal{T} \subset \mathbb{R}$ cannot be presented as a countable union of closed sets each of which has empty interior in $\mathcal{T}$.

SOLUTION:

Assume that we know that the set of algebraic numbers $\mathcal{A}$ is countable. Then

$$\mathcal{T} = \bigcap_{a \in \mathcal{A}} (\mathbb{R} \setminus \{a\})$$
is dense $G_\delta$. Hence $T$ is a Baire space by Proposition proven in class. Then $T$ cannot be presented as a countable union of closed sets each of which has empty interior by the definition.

Now we show that the set of algebraic numbers $\mathcal{A}$ is countable. Thus, each $a \in \mathcal{A}$ is a root of a polynomial with rational coefficients. We note that the set $\mathcal{P}_n$ of polynomials of degree $\leq n$ with rational coefficients is in bijection with $\mathbb{Q}^{n+1}$ and, hence, is countable. Also every set $\mathcal{P}_n \times \{1, 2, \ldots, n\}$ is countable and so is the union

$$\mathbb{P} = \bigcup_n (\mathcal{P}_n \times \{1, 2, \ldots, n\}).$$

We define an injective map $s : \mathcal{A} \to \mathbb{P}$ as follows: For each $a \in \mathcal{A}$ we fix a polynomial $p \in \mathcal{P}_n$ of some $n = n(a)$ such that $p(a) = 0$. If $a$ is the $i$-th root with respect to the order on $\mathbb{R}$ we define

$$s(a) = p \times i \in \mathcal{P}_n \times \{1, 2, \ldots, n\}.$$ 

This implies that $\mathcal{A}$ is countable.

4. Assuming the fact that $I^I$ is separable, show that the cardinality of $\beta(\mathbb{N})$ is at least as great as $I^I$ where $I = [0, 1]$.

SOLUTION: Let $f : \mathbb{N} \to I^I$ be a map onto a dense subset. By the universal property of the Stone-Čech compactification there is a continuous extension $\bar{f} : \beta(\mathbb{N}) \to I^I$. Since it is continuous and $\bar{f}(\beta(\mathbb{N}))$ is dense and compact, the extension $\bar{f} : \beta(\mathbb{N}) \to I^I$ is onto. Hence the cardinality of $\beta(\mathbb{N})$ is at least as great as $I^I$.

5. (Extra Credit) Prove that $I^I$ is separable.

SOLUTION: The set of piecewise constant functions with rational values and with finitely many rational breaking points in $I$ is countable and dense in $I^I$. 