MIDTERM SOLUTIONS

1. Show that there is no retractions \( r : X \to A \) in the following cases:

   **SOLUTION:** Let \( i : A \to X \) denote the inclusion. In case of retraction we have that \( i_* \) is injective and \( r_* \) is surjective.

   (a) \( X = \mathbb{R}^3 \) with \( A \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \) equal the unit circle.
   Since \( \pi_1(A) = \mathbb{Z} \) and \( \pi_1(\mathbb{R}^3) = 0 \), the homomorphism \( i_* \) cannot be injective. Thus, there is no retraction.

   (b) \( X = S^1 \times D^2 \) with \( A \) its boundary \( S^1 \times S^1 \).
   Since \( \pi_1(A) = \mathbb{Z} \times \mathbb{Z} \) and \( \pi_1(X) = \mathbb{Z} \), the homomorphism \( i_* \) cannot be injective. Thus, there is no retraction.

   (c) \( X = D^2 \vee D^2 \) and \( A = S^1 \vee S^1 \) its boundary.
   Since \( \pi_1(A) = \mathbb{Z} \ast \mathbb{Z} \) and \( \pi_1(X) = 0 \), the homomorphism \( i_* \) cannot be injective. Thus, there is no retraction.

   (d) \( X \) is the Mobius band and \( A \) its boundary.
   Suppose there is a retraction \( r : X \to A \). Then \( r_*i_* = 1 \). Since \( i \) is homotopic to a map tracing out the mid circle of the Mobius band twice, \( i_* : \mathbb{Z} \to \mathbb{Z} \) is multiplication by 2. Then we obtain that 1 is divisible by 2: \( 1 = r_*i_*(1) = r_*(2) = 2r_*(1) \). Contradiction.

2. Let \( X \) be the quotient space of \( S^2 \) obtained by identifying the north and the south poles to a single point. Put a CW complex structure on \( X \) and use it to compute \( \pi_1(X) \).

   **SOLUTION:** Let \( a \) be a meridian from the south pole \( S \) to the north pole \( N \). Then the complement to \( a \) is an open 2-cell. Thus, we have a CW complex structure on \( S^2 \) with two 0-cells \( S \) and \( N \), one 1-cell \( a \), and one 2-cell \( e \). When we identify \( S \) and \( N \) we obtain a CW complex structure on \( X \). The attaching map of \( e \) is represented by the word \( aa^{-1} \). To see that we cut \( S^2 \) open along \( a \) to obtain a disk with two copies of \( a \) on the boundary with the direction from \( S \) to \( N \). Thus, \( \pi_1(X) \) has a presentation \( \langle a \mid aa^{-1} \rangle \) which is after reduction \( \langle a \rangle = \mathbb{Z} \).

3. Let \( X \subset \mathbb{R}^3 \) be the union of 5 lines through the origin. Compute \( \pi_1(\mathbb{R}^3 \setminus X) \).

   **SOLUTION:** Using the radial projection we can deform \( X \) to the sphere with 10 points removed: 2 for each line. The 2-sphere with 10
points removed is homeomorphic to $\mathbb{R}^2$ with 9 points removed. The latter is homotopy equivalent to the wedge of 9 circles. Thus, $\pi_1(X) = F_9$, free group with 9 generators.

4. Construct the universal covering of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter.

Solution: It is an alternating chain of spheres and intervals.

5. (extra credit) Find all connected 3-sheeted coverings of $S^1 \vee S^1$.

Answer: There are 7 of such