1. Let $p : \tilde{X} \to X$ be a covering map. Let $x_0 \in A \subset X$ and $\tilde{x}_0 \in p^{-1}(x_0)$. Show that $p_* : \pi_n(\tilde{X}, p^{-1}(A), \tilde{x}_0) \to \pi_n(X, A, x_0)$ is an isomorphism for all $n > 2$.

**SOLUTION:** Consider the commutative diagram generated by the long exact sequences of pairs $(\tilde{X}, p^{-1}(A))$ and $(X, A)$.

\[
\begin{array}{cccccc}
\pi_n(p^{-1}(A)) & \longrightarrow & \pi_n(\tilde{X}) & \longrightarrow & \pi_n(\tilde{X}, p^{-1}(A)) & \longrightarrow & \pi_{n-1}(p^{-1}(A)) & \longrightarrow & \pi_{n-1}(\tilde{X}) \\
\cong & \downarrow & \cong & \downarrow & p_* & \downarrow & \cong & \downarrow & \cong \\
\pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(X, A) & \longrightarrow & \pi_{n-1}(A) & \longrightarrow & \pi_{n-1}(X)
\end{array}
\]

and apply Five Lemma to conclude that $p_*$ is an isomorphism.

**EXTRA Credit** $n = 2$: In the diagram

\[
\begin{array}{cccccc}
\pi_2(p^{-1}(A)) & \longrightarrow & \pi_2(\tilde{X}) & \longrightarrow & \pi_2(\tilde{X}, p^{-1}(A)) & \longrightarrow & \pi_1(p^{-1}(A)) & \longrightarrow & \pi_1(\tilde{X}) \\
\cong & \downarrow & \cong & \downarrow & p_* & \downarrow & \alpha & \downarrow & \cong \\
\pi_2(A) & \longrightarrow & \pi_2(X) & \longrightarrow & \pi_2(X, A) & \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(X)
\end{array}
\]

the homomorphism $\alpha$ is the induced by a covering map $p|_{\tilde{A}_0} : \tilde{A}_0 \to A_0$ of a path component $\tilde{A}_0$ of $p^{-1}(A)$ containing $\tilde{x}_0$ onto a path component $A_0$ of $A$. Therefore $\alpha$ is a monomorphism. By the mono version of the 5-Lemma $p_*$ is injective. Since every map $f : (D^2, s_0) \to (X, x_0)$ admits a lift $\tilde{f} : (D^2, s_0) \to (\tilde{X}, \tilde{x}_0)$, where $s_0 \in \partial D^2$, the homomorphism $p_*$ is surjective.

2. Show that a CW complex $X$ retracts onto any contractible subcomplex $A$.

**SOLUTION:** We construct a retraction $r : X^k \cup A \to A$ by induction on $k$. Inductive step follows from the fact that a map of the $k$-skeleton $X^k$ extends to a map of the $(k+1)$-dimensional skeleton $X^{k+1}$ provided $\pi_k(A) = 0$. The latter holds true, since $A$ is contractible.

3. Show that if $X$ is $m$-connected and $Y$ is $n$ connected CW complexes, then the smash product $X \wedge Y$ is $(m+n+1)$-connected.

**SOLUTION:** There are homotopy equivalences $f_X : X \to X_1$ and $f_Y : Y \to Y_1$ to CW complexes such that $X_1^m = x_0$ and $Y_1^n = y_0$. Let $g_X$ and $g_Y$ be the homotopy inverse We may assume that all these maps take the base points to the base points.

**Claim (proven in the class):** Suppose that CW complexes $X$ and $X'$ both contain a subcomplex $A$ and let $f : X \to X'$ be a homotopy
equivalence with \( f|_A = 1_A \). Then there is homotopy equivalence of pairs \( f : (X, A) \to (X', A) \).

We apply this claim to \( f_X \) and \( f_Y \) with \( A = pt \). Then the homotopy equivalence \( f_X \times f_Y : X \times Y \to X_1 \times Y_1 \) is a homotopy equivalence of the pairs \((X \times Y, X \vee Y)\) and \((X_1 \times Y_1, X_1 \vee Y_1)\). Indeed, if \( h_t : (X, x_0) \to (X, x_0) \) and \( q_t : (Y, y_0) \to (Y, y_0) \) are homotopies from \( g_X f_X \) to \( 1_X \) and \( g_Y f_Y \) to \( 1_Y \), then \((h_t \times q_t)(X \times y_0 \cup x_0' \times Y) \subset X \times y_0 \cup x_0' \times Y\). Therefore, the homotopy equivalence \( f_X \times f_Y \) defines the homotopy equivalence of the quotient spaces \( F : (X \times Y)/(X \vee Y) = X \wedge Y \to (X_1 \times Y_1)/(X_1 \vee Y_1) = X \). Since \((X_1 \wedge Y_1)^{m+n+1} = pt \), \( X_1 \wedge Y_1 \) is \((m + n + 1)\)-connected.

4. Show that the action of \( \pi_1(\mathbb{R}P^2) \) on \( \pi_2(\mathbb{R}P^2) \) is nontrivial.

SOLUTION. Let \( p : S^2 \to \mathbb{R}P^2 \) be the universal covering map, \( p(s_0) = x_0 \). There is an isomorphism \( \Phi : \pi_2(\mathbb{R}P^2, x_0) \to \pi_2(S^2, s_0) \) defined as follows: Since \( S^2 \) is simply connected, for any continuous map \( f : (S^2, s_0) \to (\mathbb{R}P^2, x_0) \) there is a unique lift \( \tilde{f} : (S^2, s_0) \to (S^2, s_0) \) of \( f \). The isomorphism \( \Phi \) takes the class \([f]\) to \([\tilde{f}]\). Let \( \tilde{\gamma} \) be a path from \( s_0 \) to \(-s_0\) and let \( \gamma = p \tilde{\gamma} \). Then \( \gamma \) generates \( \pi_1(\mathbb{R}P^2, x_0) \). We show that \( \gamma[p] \neq [p] \). For that we show that \( \Phi(\gamma[p]) \neq \Phi([p]) \). We use the degree isomorphism \( deg : \pi_2(S^2, s_0) \to \mathbb{Z} \). Note that \( \Phi([p]) = [1_{S^2}] \) is the identity class. Hence \( deg(\tilde{p}) = 1 \). Note that the lift of \( \gamma^* p : I^2 \to \mathbb{R}P^2 \) is \( \tilde{\gamma}^* p' \) where \( p' : (S^2, s_0) \to (S^2, -s_0) \) is the unique lift of \( p \) with the initial point \(-s_0\). Thus \( p' = -1 \) is the antipodal map. Note that the degree of \( \tilde{\gamma}^* p' \) equals the degree of \( p' \), equals -1.

5. Show that any \( n \)-connected \( n \)-dimensional CW complex \( X \) is contractible.

SOLUTION: We construct by induction on \( k \) a map \( H : X^k \times [0, 1] \cup X \times \{0, 1\} \to X \) with the restriction to \( X \times 0 \) and \( X \times 1 \) equal to the identity and a constant map respectively. Since \( X \) is \( n \)-connected we can do it up to \( k = n \). Note that \( X^n = X \).