Topology, MTG 4303/5317 Spring-2026 SOLUTIONS:

Quiz 1

1. Let Y be a compactification of X. Show that there is a continuous map $g : \beta(X) \to Y$ of the Stone-Čech compactification onto Y such that $g(x) = x$ for all $x \in X$.

SOLUTION: Let $i : X \to Y$, $i(x) = x$, be the inclusion. Since Y is compact Hausdorff and i is continuous, by the main Property of $\beta(X)$ there a continuous extension $g : \beta(X) \to Y$. Since g is an extension of i, we obtain $g(x) = i(x) = x$.

2. Show that compact absolute retract has the universal extension property.

RECALL definitions:

A space Y has the *universal extension property* if for each closed subset $A \subset X$ of a normal space X and a continuous map $f : A \to Y$ there is a continuous extension $\bar{f} : X \to Y$.

A normal space Y is called an *absolute retract* if for every embedding $g : Y \to Z$ to a normal space as a closed subset there is a retraction $r : Z \to g(Y)$.

SOLUTION: Let Y be an absolute retract. Since Y is normal (by definition), it can be embedded to a Tychonoff cube $[0,1]^J$. Let $g : Y \to [0,1]^J$ be an embedding. Since Y is compact, the image $g(Y)$ is compact and, hence, is closed in $[0,1]^J$. Then there is a retraction $r : [0,1]^J \to g(Y)$.

Let $A \subset X$ be a normal space X and let $f : A \to Y$ be a continuous map. By the Tietze Extension Theorem the map $\pi_\alpha g f : A \to [0,1]$ has a continuous extension $h_\alpha : X \to [0,1]$ where $\pi_\alpha : [0,1]^J \to [0,1]$ is the projection onto the factor in the product $[0,1]^J$ indexed by $\alpha \in J$. The family $(h_\alpha)_{\alpha \in J}$ defines a continuous map $h : X \to [0,1]^J$. Let $g^{-1} : g(Y) \to Y$ be the inverse map for the homeomorphism $g : Y \to g(Y)$. Then the map $\bar{f} = g^{-1} r h : X \to Y$ is well-defined continuous map extending f: For $a \in A$ we have

$$\bar{f}(a) = g^{-1} r h(a) = g^{-1} r g(a) = g^{-1} g(a) = a.$$

Here we used the facts that $h_\alpha(a) = \pi_\alpha g(a)$ for all $\alpha \in J$, hence $h(a) = g(a)$, and $r(g(a)) = g(a)$ by the definition of retraction.