

Quiz 1

1. *Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.*

Let $f_0, f_1 : X \rightarrow Y$ be homotopic maps via homotopy f_t . Let f_0 be a homotopy equivalence with a homotopy inverse $g : Y \rightarrow X$. We show that g is a homotopy inverse for f_1 as well. Indeed, $f_1 g$ is homotopic to $f_0 g \sim 1_Y$ via the homotopy $f_t g$. Hence $f_1 g \sim 1_Y$ by transitivity. Similarly, $g f_1$ is homotopic to $g f_0 \sim 1_X$ by means of the homotopy $g f_t$ and, hence, $g f_1 \sim 1_X$.

2. *Show that the space obtained from S^2 by attaching n 2-cells along any collection of n circles is homotopy equivalent to the wedge of $n + 1$ 2-spheres.*

We note that each 2-cell is attached to S^2 by means of a nullhomotopic map $\phi_i : S^1 \rightarrow S^2$, $i = 1, \dots, n$.

We recall that Theorem 3 (lectures) states: *The space $X_f = X_0 \cup_f X_1$ is homotopy equivalent to the space $X_g = X_0 \cup_g X_1$ where $f, g : A \rightarrow X_0$ are homotopic maps and (X_1, A) is a CW pair.*

First we note that the union of the attaching maps

$$f = \cup \phi_i : \coprod_{i=1}^n S^1 \rightarrow S^2$$

is homotopic to a constant map

$$g : \coprod_{i=1}^n S^1 \rightarrow x_0 \in S^2.$$

Then we can apply Theorem 2 with

$$X_0 = S^2, \quad X_1 = \coprod_{i=1}^n D_i^2, \quad A = \coprod_{i=1}^n S_i^1, \quad \text{and the maps } f \sim g$$

to obtain homotopy equivalent spaces X_f and X_g . Note that the space X_f is the space in question and the space X_g is the wedge of $n + 1$ spheres with x_0 the common point.