MTG 4302/5316 Fall-2019 Quiz 1, SOLUTIONS:

Which of the following subsets of $\mathbb{R}^{\omega}$ can be expressed as the cartesian product of subsets of $\mathbb{R}$? Justify your answer.

Note that $\mathbb{R}^{\omega} = \prod_{i \in \mathbb{N}} \mathbb{R}_i$ where $\mathbb{R}_i = \mathbb{R}$ for all $i$. Then

(a) $\{x \mid x_i \in \mathbb{Z} \ \forall i\} = \prod_{i \in \mathbb{N}} \mathbb{Z}_i$ where $\mathbb{Z}_i = \mathbb{Z} \subset \mathbb{R}$.

(b) $\{x \mid x_i \geq i \in \mathbb{Z} \ \forall i\} = \prod_{i \in \mathbb{N}} X_i$ where $X_i = [i, \infty) \subset \mathbb{R}$.

(c) $\{x \mid x_i \in \mathbb{Z} \ \forall i \geq 100\} = \prod_{i \in \mathbb{N}} X_i$ where $X_i = \mathbb{R}$ for $i < 100$ and $X_i = \mathbb{Z}$ for $i \geq 100$.

(d) This is a tricky question. The answer depends on what to understand under words 'can be expressed'.

The set $D = \{x \mid x_2 = x_3\}$ is not a sub-product of the product $\prod_{i \in \mathbb{N}} \mathbb{R}_i$ since for any subproduct $X = \prod_{i \in \mathbb{N}} X_i$, $X_i \subset \mathbb{R}_i$ for each $i$ the equality $\pi_i(X) = X_i$ holds where $\pi_i : \prod_{i \in \mathbb{N}} \mathbb{R}_i \to \mathbb{R}_i$ is the projection onto the $i$-th factor. Note that $\pi_i(D) = \mathbb{R}_i$ for all $i$ but $D$ is a proper subset of $\prod_{i \in \mathbb{N}} \mathbb{R}_i$.

On the other hand the set $D$ has the following product structure $\{x \mid x_2 = x_3\} = \mathbb{R} \times \Delta \times \prod_{i=4}^{\infty} \mathbb{R}_i$ where $\Delta$ is the diagonal in the plane $\mathbb{R}_2 \times \mathbb{R}_3$. Since the diagonal is isomorphic to $\mathbb{R}$, we can say that $D$ is expressed as a product of subsets of $\mathbb{R}$.