Topology 2 QUIZ 1 NAME:

1. Let $X = (S^1 \times S^1) \cup_f M$ where M is the Mobius band and $f : \partial M \to S^1 \times S^1$ is a homeomorphism onto $S^1 \times \{x_0\}$. Compute the homology groups of X.

SOLUTION. By the Mayer-Vietoris exact sequence for $X = T \cup M$ with $T \cap M = S^1$ we have

$$0 \to H_2(T) \oplus H_2(M) \to H_2(X) \to H_1(S^1) \xrightarrow{\phi} H_1(T) \oplus H_1(M) \to H_1(X) \to 0.$$

We know that $H_2(T) = \mathbb{Z}$, $H_2(M) = 0$, $H_1(S^1) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}$, and $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$. The homomorphism ϕ is defined as $\phi(x) = (j^1_*(x), -j^2_*(x))$ where $j^1 : S^1 \to T$ and $j^2 : S^1 \to M$ are the inclusions. Since the projection $T \to S^1$ is a retraction, the homomorphism j^1_* is injective. Hence ϕ is injective. Therefore, $H_2(X) = H_2(T) \oplus H_2(M) = \mathbb{Z}$.

The homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}^3$ takes the generator to $(1, 0, -2)$ which is a basis vector for the basis $(1, 0, -2)$, $(0, 1, 0)$, $(0, 0, 1)$.

Hence $H_1(X) = coker(\phi) = \mathbb{Z}^3/\mathbb{Z} = \mathbb{Z}^2$.

2. Show that $f : S^n \to S^n$ has a fixed point unless its degree equals the degree of the antipodal map.

SOLUTION. The degree of the antipodal map equals $(-1)^{n+1}$. Thus, $deg(f) \neq (-1)^{n+1}$. The Lefschetz number $\tau(f) = 1 + deg(f)$ if n even and $\tau(f) = 1 + deg(f)$ if n is odd. In both cases $\tau(f) \neq 0$. By the Lefschetz Fixed Point Theorem there is a fixed point.