1. Compute $H_i(S^n \setminus X)$ where $X$ is a subspace of $S^n$ homeomorphic to $S^k \vee S^\ell$.

SOLUTION. Note that the complement $(S^k \vee S^\ell)^c = (S^k)^c \cap (S^\ell)^c$. Consider the Mayer-Vietoris exact sequence for reduced homology applied to $(S^k)^c \cup (S^\ell)^c = (S^k \cap S^\ell)^c \cong \mathbb{R}^n$.

$$
\cdots \to \tilde{H}_{i+1}((S^k)^c \cup (S^\ell)^c) \to \tilde{H}_i((S^k)^c \cap (S^\ell)^c) \to \tilde{H}_i((S^k)^c) \oplus \tilde{H}_i((S^\ell)^c) \to \tilde{H}_i((S^k)^c \cup (S^\ell)^c) \to \cdots
$$

Since $\tilde{H}_i((S^k)^c \cup (S^\ell)^c) = 0$ for all $i$, we obtain

$$\tilde{H}_i(S^n \setminus X) = \tilde{H}_i((S^k)^c) \oplus \tilde{H}_i((S^\ell)^c).$$

Thus, if $k \neq \ell$,

$$H_i(S^n \setminus X) = \begin{cases} 
\mathbb{Z} & i = 0, \; i = n - k - 1 \text{ or } i = n - \ell - 1 \\
0 & \text{otherwise}
\end{cases}$$

If $k = \ell$,

$$H_i(S^n \setminus X) = \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z} & i = 0, \; \text{or } i = n - k - 1 \\
0 & \text{otherwise}
\end{cases}.$$

2. Let $M$ be a closed orientable surface embedded in $\mathbb{R}^3$ in such a way that reflection across a plane $P$ defines a homeomorphism $r : M \to M$ fixing $M \cap P$, a collection of circles. Is it possible to homotope $r$ to have no fixed points?

SOLUTION. First we define a homeomorphism $h : S^1 \times [-1, 1] \to S^1 \times [-1, 1]$ such that $h$ is the identity on the boundary, It is the rotation on angle $\alpha \neq 0$ on the mid circle $S^1 \times \{0\}$ and it is rotation to the angle $(1 - |t|)\alpha$ on the circle $S^1 \times \{\pm t\}$, $t \in [-1, 1]$. Note that $h$ is homotopic to the identity rel $S^1 \times \{\pm 1\}$.

The answer to the question is Yes. Rotate the circles $M \cap P$ and extend rotation inside. For that consider a band $N \cong S^1 \times I$ around each circle and apply the homeomorphism $h$. Extend $h$ to a homeomorphism $\tilde{h} : M \to M$ by the identity. Note that $\tilde{h}$ is homotopic to the identity. Thus, $\tilde{h} \circ r : M \to M$ is homotopic to $r$ and does not have fixed points.