Topology 2 QUIZ 2 NAME:

1. Compute the simplicial cohomology $H^*(K;\mathbb{Z}_2)$ of the Klein bottle using a $\Delta$-complex structure on $K$.

SOLUTION: The chain complex for the standard $\Delta$-complex structure for $K$ is

$$0 \to \mathbb{Z}\langle U, L\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle a,b,c\rangle \xrightarrow{0} \mathbb{Z}\langle v\rangle \to 0$$

with $\partial_1(U) = b - c + a$ and $\partial_1(L) = a - c + b$. Let $a^* : Z\langle a, b, c, \rangle \to \mathbb{Z}_2$ be the dual to $a$, i.e., $a^*(a) = 1$ and $a^*(b) = a^*(c) = 0$. similarly we define $b^*$, $c^*$, $U^*$, and $L^*$. Then the cochain complex is

$$0 \leftarrow \mathbb{Z}_1\langle U^*, L^*\rangle \xleftarrow{\delta_1} \mathbb{Z}_2\langle a^*, b^*, c^*\rangle \xleftarrow{0} \mathbb{Z}\langle v^*\rangle \leftarrow 0.$$

Then $H^0(K;\mathbb{Z}_2) = \mathbb{Z}_2$, $H^1(K;\mathbb{Z}_2) = ker(\delta_1)$, and $H^2(K;\mathbb{Z}_2) = coker(\delta_1)$.

Note that $\delta_1(a^*) = U^* + L^* = \delta_1(b^*) = \delta_1(c^*)$. Hence $coker(\delta_1) = \mathbb{Z}_2$ and $ker(\delta_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

2. Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

SOLUTION: We know that the cohomology ring

$$H^*(\mathbb{R}P^3;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^4)$$

whereas the cohomology ring

$$H^*(\mathbb{R}P^2 \vee S^3;\mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^3) \oplus \mathbb{Z}_2[\gamma]/(\gamma^2).$$

Since $x^3 = 0$ for all $x$ in the former and $\alpha^3 \neq 0$ in the latter, these rings cannot be isomorphic. Hence, the spaces are not homotopy equivalent.