MTG 4303/5317 Spring2026 SOLUTIONS:

1. Let $[X, Y]$ denote the set of homotopy classes of maps of X to Y. A space X is contractible if $[X, X]$ has a single element.

(a) Show that contractible space is path connected.

(b) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

(c) Show that if Y is contractible, then $[X, Y]$ has a single element for any X.

SOLUTION.

(a) *Solution 1:* Since $[X, X]$ has a single element, any two maps $f, g : X \to X$ are homotopic. Then for any two points $x, y \in X$ the constant maps e_x and e_y are homotopic. Let $H : X \times [0, 1] \to X$ be a homotopy between e_x and e_y. Then $f(t) = H(x, t)$ is a path from $f(0) = H(x, 0) = e_x(x) = x$ to $f(1) = H(x, 1) = e_y(x) = y$.

Solution 2: Let $H : X \times [0, 1] \to X$ be a homotopy between the idenity map $i : X \to X$ and the constant map $e_{x_0} : X \to X$ to $x_0 \in X$. Then for any $x \in X$ the map $f(t) = H(x, t)$ is a path from x to x_0. Therefore every $x \in X$ lies in the path componet $P(x_0)$ of x_0. Thus, the space $X = P(x_0)$ is path connected.

(b) Let $y, y' \in Y$ and let $f : [0, 1] \to Y$ be a path from y to y'. Then the constant maps $e_y : X \to Y$ and $e_{y'} : X \to Y$ are homotopic via the homotopy $H(x, t) = f(t)$. Since Y is path connected this holds true for all $y, y' \in Y$. It means all constant maps $X \to y$ are homotopic.

Since X is contractible, the identity map i is homotopic to a constant map $\bar{e}_{x_0} : X \to X$ via some homotopy $G : X \times [0, 1] \to X$ then for any map $f : X \to Y$ we have a homotopy $f \circ G$ between $f \circ i = f$ and $f \circ \bar{e}_{x_0} = e_{f(x_0)}$. Thus all maps $f : X \to Y$ are homotopic to constant maps, and, hence, are homotopic.

(c) Let $H : Y \times [0, 1] \to y$ be a homotopy of the identity map to a constant map e_{y_0} Then any map $f : X \to Y$ is homotopic to the constant map via the homotopy $F(x, t) = H(f(y), t)$.

2. Let $p : E \to B$ be a covering map for connected space B. Show that if the cardinality of $p^{-1}(b_0)$ equal $k \in \mathbb{N}$, then $|p^{-1})b)| = k$ for every $b \in B$.

SOLUTION:

Let $U = \{x \in b \mid |p^{-1}(x)| = k\}$. Clearly, the set U is open. Let $V = \{x \in b \mid |p^{-1}(x)| \neq k\}$. We show that V is open. Let $x \in V$ and let U be an even neighborhood of x. Then $p^{-1}(U) = \coprod_{i \in J} U_i$ Since $x \in U$, we have $|J| \neq k$.

If $V \neq \emptyset$ we have a separation $B = U \sqcup V$ which contradicts to connectivity of B.